1.7 #12 For each of the following conditions, give an example of an unsatisfiable set $\Gamma$ of formulas that meets the condition.

(a) Each member of $\Gamma$ is - by itself - satisfiable.

**Solution:** Let $\gamma_1 = A_1$ and $\gamma_2 = \neg A_1$ and $\Gamma = \{\gamma_1, \gamma_2\}$.

(b) For any two members $\gamma_1, \gamma_2$ of $\Gamma$, the set $\{\gamma_1, \gamma_2\}$ is satisfiable.

**Solution:** Let $\gamma_1 = A_1$, $\gamma_2 = \neg A_2$, $\gamma_3 = \neg A_1 \lor A_2$ and $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$. Then the two element subsets of $\Gamma$ are

- $\{A_1, \neg A_2\}$: satisfiable by $v(A_1) = T, v(A_2) = F$.
- $\{A_1, \neg A_1 \lor A_2\}$: satisfiable by $v(A_1) = T, v(A_2) = T$.
- $\{\neg A_2, \neg A_1 \lor A_2\}$: satisfiable by $v(A_1) = F, v(A_2) = F$.

But, no truth assignment simultaneously satisfies all formulas in $\Gamma$.

1.7* A Prove that soundness is equivalent to the statement “every satisfiable set of formulas is consistent.”

**Solution:** Recall that soundness is the statement: for all sets of wffs $\Sigma$ and $\varphi$ a wff, $\Sigma \vdash \varphi$ implies $\Sigma \models \varphi$.

Assume soundness. We will prove the contrapositive: if $\Sigma$ is inconsistent then it is not satisfiable. Suppose $\Sigma$ is inconsistent and hence there is a wff $\varphi$ such that $\Sigma \vdash \varphi$ and $\Sigma \vdash \neg \varphi$. Soundness implies that $\Sigma \models \varphi$ and $\Sigma \models \neg \varphi$. Let $v$ be a truth assignment that satisfies every wff in $\Sigma$. By above, $\bar{v}(\varphi) = \bar{v}(\neg \varphi) = T$. This is impossible. Hence, no truth assignment satisfies every wff in $\Sigma$ and $\Sigma$ is unsatisfiable.

Conversely, suppose every satisfiable set of formulas is consistent and let $\Sigma, \varphi$ be as stipulated. We work towards a proof by contradiction: assume $\Sigma \vdash \varphi$ but $\Sigma \not\vdash \varphi$. Then by fact proved in class and on page 60 of Enderton, $\Sigma \cup \{\neg \varphi\}$ is satisfiable. Therefore (by hypothesis), $\Sigma \cup \{\neg \varphi\}$ is consistent. Since $\Sigma \vdash \varphi$, $\Sigma \cup \{\neg \varphi\} \vdash \varphi$. Also, since $\neg \varphi \in \Sigma \cup \{\neg \varphi\}$, $\Sigma \cup \{\neg \varphi\} \vdash \neg \varphi$. This contradicts the consistency of $\Sigma \cup \{\neg \varphi\}$.

1.7* B Prove that completeness is equivalent to the statement “every consistent set of formulas is satisfiable”.

**Solution:** Recall that completeness is the statement: for all sets of wffs $\Sigma$ and $\varphi$ a wff, $\Sigma \models \varphi$ implies $\Sigma \vdash \varphi$.

Assume completeness. Towards a contradiction, suppose $\Sigma$ is a consistent set of formulas that is not satisfiable. Since it is not satisfiable, it tautologically implies anything. In particular, $\Sigma \vdash A_1 \land \neg A_1$. By completeness assumption, this implies that $\Sigma \vdash A_1 \land \neg A_1$ and let $\langle \alpha_0, \ldots, \alpha_n \rangle$ be a deduction that witnesses this. Note that $A_1 \land \neg A_1 \rightarrow A_1$ and $A_1 \land \neg A_1 \rightarrow \neg A_1$ are both tautologies (can verify by truth-table). Then

$$\langle \alpha_0, \ldots, \alpha_n, A_1 \land \neg A_1 \rightarrow A_1, A_1 \rangle$$

is a deduction that witnesses $\Sigma \vdash A_1$ and

$$\langle \alpha_0, \ldots, \alpha_n, A_1 \land \neg A_1 \rightarrow \neg A_1, \neg A_1 \rangle$$

is a deduction that witnesses $\Sigma \vdash \neg A_1$. This contradicts the assumption that $\Sigma$ is consistent.

Conversely, suppose any consistent set is satisfiable. We will prove the contrapositive: that is, assume $\Sigma \not\models \varphi$ and we will show that $\Sigma \not\vdash \varphi$. By Part (b) of fact proved in lecture, $\Sigma \cup \{\neg \varphi\}$ is consistent. The assumption then implies that $\Sigma \cup \{\neg \varphi\}$ is satisfiable. That is, there is a truth assignment which satisfies every wff in $\Sigma$ and $\neg \varphi$. This assignment satisfies every wff in $\Sigma$ and does not satisfy $\varphi$. Therefore, $\Sigma \not\vdash \varphi$. 

2.1 #2 Translate into good English the wff
\[ \forall x (Nx \rightarrow Ix \rightarrow \neg \forall y (Ny \rightarrow Iy \rightarrow \neg x < y)). \]

We add in some parentheses dropped according to convention 4 from page 78:
\[ \forall x (Nx \rightarrow [Ix \rightarrow \neg \forall y (Ny \rightarrow (Iy \rightarrow \neg x < y))]). \]

Recall that \( \exists x \alpha \) abbreviates \( \neg \forall x (\neg \alpha) \) and therefore:
\[ \forall x (Nx \rightarrow [Ix \rightarrow \exists y (Ny \rightarrow (Iy \rightarrow \neg x < y))]). \]

This is tautologically equivalent to
\[ \forall x (Nx \rightarrow [Ix \rightarrow \exists y (Ny \land (Iy \land x < y))]). \]

Thus, all interesting numbers have some interesting number above them.

2.1 # 3 Translate into the first-order language specified: Neither \( a \) nor \( b \) is a member of every set. (\( \forall \), for all sets; \( \in \), is a member of; \( a, a; b, b \).
\[ \neg (\forall x (a \in x) \lor (\forall x (b \in x))) \]