(1) Question 2.4 # 4 Let $\Gamma = \{\neg \forall v_1 P v_1, P v_2, P v_3, \ldots \}$. Is $\Gamma$ consistent? Is $\Gamma$ satisfiable?

Solution: By soundness and completeness, $\Gamma$ is consistent if and only if it is satisfiable. Therefore, it suffices to prove either one. In particular, we give a structure and an assignment which satisfies every formula in $\Gamma$. Consider $\mathfrak{A}$ with $|\mathfrak{A}| = \{a, b\}$ and $P^\mathfrak{a} = \{a\}$ and the assignment $s(v_i) = a$ for all $i$. Then

$$\models_\mathfrak{A} \neg \forall v_1 P v_1[s]$$

because it is not the case that $P^\mathfrak{a}$ holds of $b$. But, for each $i \geq 2$,

$$\models_\mathfrak{A} P v_i[s]$$

because $P^\mathfrak{a}$ holds of $s(v_i)$, namely of $a$.

(2) Question 2.4 # 6 Let $\Sigma_1$ and $\Sigma_2$ be sets of sentences such that nothing is a model of both $\Sigma_1$ and $\Sigma_2$. Show that there is a sentence $\tau$ such that $\text{Mod}\Sigma_1 \subseteq \text{Mod}\tau$ and $\text{Mod}\Sigma_2 \subseteq \text{Mod}\neg \tau$. Suggestion: $\Sigma_1 \cup \Sigma_2$ is unsatisfiable; apply compactness.

Solution: Following the suggestion, consider $\Sigma_1 \cup \Sigma_2$. By assumption this is an unsatisfiable set of sentences. Therefore, the compactness theorem gives that some finite subset of it is unsatisfiable. Call this finite unsatisfiable set $\Sigma_0$. We can write

$$\Sigma_0 = \{\varphi_1, \ldots, \varphi_n\} \cup \{\psi_1, \ldots, \psi_m\}$$

where each $\varphi_i \in \Sigma_0 \cap \Sigma_1$ and each $\psi_j \in \Sigma_0 \setminus \Sigma_1$. Let

$$\tau = (\varphi_1 \land \cdots \land \varphi_n).$$

To prove that this $\tau$ works:

- Suppose $\mathfrak{A} \in \text{Mod}\Sigma_1$. That is, for every $\varphi \in \Sigma_1$, $\models_\mathfrak{A} \varphi$. Therefore $\models_\mathfrak{A} \varphi_1, \models_\mathfrak{A} \varphi_2$, etc. By definition of satisfaction of conjunctions, $\models_\mathfrak{A} \tau$. Thus, $\mathfrak{A} \in \text{Mod}\tau$.
- Suppose, for a contradiction, that there is some structure $\mathfrak{A}$ such that $\mathfrak{A} \in \text{Mod}\Sigma_2$ but $\mathfrak{A} \notin \text{Mod}\neg \tau$. That is, for each $\psi \in \Sigma_2 \models_\mathfrak{A} \psi$ and it is also the case that $\not\models_\mathfrak{A} \neg \tau$. By definition of satisfaction for negated formulas, we have that $\models_\mathfrak{A} \tau$ and hence $\models_\mathfrak{A} \varphi_i$ for each $i$. Since each of the $\psi_j$ in $\Sigma_0$ is a member of $\Sigma_2$, $\models_\mathfrak{A} \psi_j$ for each $j$. Thus, $\mathfrak{A}$ satisfies each of the formulas in $\Sigma_0$. But, by assumption $\Sigma_0$ is unsatisfiable, a contradiction. Therefore, $\text{Mod}\Sigma_2 \subseteq \text{Mod}\neg \tau$.

(3) Prove that for any structure $\mathfrak{A}$ for the language of set theory ($\forall, \in$)

$$\models_\mathfrak{A} \forall x \forall y (x = y \rightarrow \forall z \in x \leftrightarrow z \in y).$$

Solution: Let $\mathfrak{A}$ be such a structure and consider arbitrary $a, b \in |\mathfrak{A}|$. Then by definition of satisfaction,

$$\models_\mathfrak{A} (x = y \rightarrow \forall z \in x \leftrightarrow z \in y)[(x[a])(y[b])]$$

if and only if $\not\models_\mathfrak{A} x = y[(x[a])(y[b])]$ or $\models_\mathfrak{A} \forall z \in x \leftrightarrow z \in y][(x[a])(y[b])]$. So suppose $\models_\mathfrak{A} x = y[(x[a])(y[b])]$ and we will show that this implies that $\models_\mathfrak{A} \forall z \in x \leftrightarrow z \in y][(x[a])(y[b])]$. By definition of satisfaction for atomic formulas, the hypothesis is that $a = b$. Therefore, we need to show that

$$\models_\mathfrak{A} \forall z \in x \leftrightarrow z \in y][(x[a])(y[a])].$$

Let $c \in |\mathfrak{A}|$ be arbitrary. We will show that $\models_\mathfrak{A} (z \in x \leftrightarrow z \in y)[(x[a])(y[a])(z[c])]$. By definition of satisfaction this is equivalent to $e^\mathfrak{a}$ holding of the pair $(a, c)$ if and only
if it holds of the pair \((a,c)\). This "if and only if" is clearly true, so we are done.

(4) **Question Mx3.1** For each nonempty set \(\mathcal{E}\) and for each \(X \in \mathcal{E}\), we define the intersection of \(\mathcal{E}\) via \(X\) by

\[
\bigcap_{X} \mathcal{E} = \{x \in X : \forall U \in \mathcal{E}(x \in U)\}
\]

Show that for any two \(X,Y\) in \(\mathcal{E}\), \(\bigcap_{X} \mathcal{E} = \bigcap_{Y} \mathcal{E}\). Show also that \(A \cap B = \bigcap\{A,B\}\).

**Solution:** Let \(X,Y\) be members of \(\mathcal{E}\). By extensionality, to show that \(\bigcap_{X} \mathcal{E} = \bigcap_{Y} \mathcal{E}\) we prove that for any set \(z\), \(z \in \bigcap_{X} \mathcal{E}\) if and only if \(z \in \bigcap_{Y} \mathcal{E}\). Suppose \(z \in \bigcap_{X} \mathcal{E}\). By definition, this means that \(z \in X\) and for all \(U \in \mathcal{E}\), \(z \in U\). In particular, when \(U = Y\), we get that \(z \in Y\). Therefore, \(z\) also satisfies the conditions necessary for membership in \(\bigcap_{Y} \mathcal{E}\). The proof of subset inclusion in the other direction is exactly the same, with the roles of \(X\) and \(Y\) flipped.

For the second part, we will show that for any set \(z\), \(z \in A \cap B\) if and only if \(z \in \bigcap\{A,B\}\). Suppose \(z \in A \cap B\). By the definition on page 24, \(A \cap B = \{x \in A : x \in B\}\). Therefore, \(z \in A\) and \(z \in B\). By the first part of this question, \(\bigcap\{A,B\} = \bigcap\{A\} = \bigcap\{B\}\) and to show membership in this set means to show membership in both \(A\) and \(B\). But, \(z\) satisfies this condition. For the reverse subset inclusion, suppose \(z \in \bigcap\{A,B\}\). Then for each \(U \in \{A,B\}, z \in U\). Namely, \(z \in A\) and \(z \in B\). But, this implies that \(z \in \{x \in A : x \in B\} = A \cap B\).

(5) **Question Mx3.2** For any two sets \(A,B\), determine whether each of the following classes is or is not a set.

1. \(\{\emptyset, x : x \in A\}\)
   
   **SET** because a definable subset of the powerset of \(A\). Namely, first apply powerset axiom to \(A\), then use the comprehension (separation) axiom to define the set containing all and only those sets in \(\mathcal{P}(A)\) with exactly two elements
   
   \[B = \{y \in \mathcal{P}(A) : \forall u \forall v \forall w (u \in y \land v \in y \land w \in y \rightarrow (u = v \lor u = w \lor v = w)) \land \exists u \exists v (u \in y \land v \in y \land u \neq v)\} \]

   Then, use comprehension axiom to refine \(B\) to just those sets such that one of their elements is the emptyset:
   
   \[\{z \in B : \exists w (w \in z \land w = \emptyset)\}\].

2. \(\{x : \text{Set}(x) \land x \neq \emptyset\}\)
   
   **NOT A SET** because if it were, applying the union axiom to it and \(\emptyset\) would give a set containing all and only sets, which is not possible.

3. \(\{x,y : x \in A \land y \in B\}\)
   
   **SET** because can be formed by comprehension from the powerset of the union of \(A\) and \(B\). That is, first form \(C = \bigcup\{A,B\}\) which is the set of all sets that are either in \(A\) or in \(B\). As in part (1.), form the powerset \(\mathcal{P}(C)\) and then restrict it to sets containing exactly two elements.

   \[D = \{\{x,y\} : x \in C, y \in C\}\].

   Applying comprehension (with parameters!) to \(D\), we can restrict this set to only include those sets one of whose elements belongs to \(A\) and the other to \(B\):

   \[\{z \in D : \exists x \exists y (x \in z \land x \in A \land y \in z \land y \in B \land \forall w (w \in z \rightarrow (w = x \lor w = y)))\}\].