Math 160A: Soundness and Completeness for Sentential Logic

Proof system for Sentential Logic.

Definition (Ex 1.7.5 p. 66). For Σ a set of wffs, define a deduction from Σ to be a finite sequence $\langle \alpha_0, \ldots, \alpha_n \rangle$ of wffs such that for each $k \leq n$, either

(a) $\alpha_k$ is a tautology, or
(b) $\alpha_k \in \Sigma$, or
(c) for some $i$ and $j$ less than $k$, $\alpha_i$ is $\alpha_j \rightarrow \alpha_k$ Modus ponens.

This sequence is a deduction for $\alpha$ if $\alpha_n = \alpha$. If there is a deduction for $\alpha$ from $\Sigma$, we say $\alpha$ is provable from $\Sigma$ and we write $\Sigma \vdash \alpha$.

Example. If $\Sigma = \{ \neg A_2, A_1 \lor A_2 \}$ then $\Sigma \vdash A_1$.

To prove this, we exhibit the deduction:

$$(\alpha_0) \ (A_1 \lor A_2) \rightarrow (\neg A_2 \rightarrow A_1)$$
$$(\alpha_1) \ (A_1 \lor A_2)$$
$$(\alpha_2) \neg A_2 \rightarrow A_1$$
$$(\alpha_3) \neg A_2$$
$$(\alpha_4) \ A_1$$

where $\alpha_1, \alpha_3 \in \Sigma$ so allowed by (b), $\alpha_2$ is the result of modus ponens, and $\alpha_0$ is a tautology, as demonstrated by the truth table

<table>
<thead>
<tr>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$(A_1 \lor A_2) \rightarrow (\neg A_2 \rightarrow A_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
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Note: this proof system allows all tautologies as axioms. There are other systems that try to be more modest about the axioms assumed and thus model natural deduction more closely. See Wikipedia for examples.

Lemma (Finiteness of proofs). For any finite nonempty set of wffs $\Sigma_0 = \{ \varphi_1, \ldots, \varphi_n \}$ and wff $\alpha$, if $\Sigma_0 \vdash \alpha$ then let

$$\sigma = \varphi_n \rightarrow (\varphi_{n-1} \rightarrow (\cdots (\varphi_1 \rightarrow \alpha) \cdots))$$

and claim

$$\vdash \sigma \quad \text{and} \quad \Sigma_0 \vdash \alpha.$$

Example. In example above, $\varphi_1 = \neg A_2$, $\varphi_2 = A_1 \lor A_2$ and

$$\sigma = (A_1 \lor A_2) \rightarrow (\neg A_2 \rightarrow \alpha).$$

Proof of Lemma. First we prove $\vdash \sigma$: we proceed by induction on the size of $\Sigma_0$.

- (Base case) If $|\Sigma_0| = 1$ then $\Sigma_0 = \{ \varphi \}$ and $\sigma = \varphi \rightarrow \alpha$. Suppose $\Sigma_0 \vdash \alpha$, that is, $\varphi \vdash \alpha$. Let $v$ be any truth assignment. Then

$$\tilde{v}(\sigma) = \tilde{v}(\varphi \rightarrow \alpha) = T$$
because if $\bar{v}(\varphi) = T$ then by $\varphi \models \alpha$, it must be that $\bar{v}(\alpha) = T$; and if $\bar{v}(\varphi) = F$ then vacuously true. Since this argument holds for all truth assignments, $\models \sigma$.

- (Inductive step) Suppose true for all nonempty sets of size $n$ and let $\Sigma_0$ be such that $\Sigma_0 = \{ \varphi_1, \ldots, \varphi_{n+1} \}$. Define $\sigma_i = \varphi_i \rightarrow (\varphi_{i-1} \rightarrow (\cdots (\varphi_1 \rightarrow \alpha) \cdots))$ for $i = 1 \ldots n+1$. Then

$$\sigma = \varphi_{n+1} \rightarrow \sigma_n.$$  

Either $\{ \varphi_1, \ldots, \varphi_n \} \models \alpha$ or not.

(Yes) Then by the inductive hypothesis $\models \sigma_n$. For $v$ any truth assignment, $\bar{v}(\sigma_n) = T$ so $\bar{v}(\sigma) = \bar{v}(\varphi_{n+1} \rightarrow \sigma_n) = T$. Thus, $\models \sigma$.

(No) Suppose (for a contradiction) that $\bar{v}(\sigma_n) = F$. By definition of $\sigma_n$, $\bar{v}(\varphi_1) = \cdots = \bar{v}(\varphi_n) = T$ and $\bar{v}(\alpha) = F$. Then this $v$ satisfies all formulas in $\Sigma_0$ but not $\alpha$, contradicting $\Sigma_0 \models \alpha$.

Consider the deduction $\langle \sigma, \varphi_1, \ldots, \varphi_n, \sigma_{n-1}, \sigma_{n-2}, \ldots, \sigma_1, \alpha \rangle$. This is a deduction because its first formula is a tautology (by first part of Lemma), the next $n$ formulas are in $\Sigma$, and the next $n$ formulas (including the last, $\alpha$) are the result of modus ponens. Thus, $\Sigma_0 \models \alpha$. $\square$

### Soundness and Completeness of Sentential Logic.

**Theorem** (Soundness of sentential logic). If $\Sigma \models \alpha$ then $\Sigma \models \alpha$.

(Only prove wffs that tautologically follow from $\Sigma$.)

**Theorem** (Completeness of sentential logic). If $\Sigma \models \alpha$ then $\Sigma \models \alpha$.

(All tautological consequences of $\Sigma$ are provable from $\Sigma$.)

**Proof of soundness.** Fix $\Sigma$. If $\Sigma \models \alpha$, there is a deduction witnessing this. Therefore, we will prove soundness by induction on the length of deductions. Namely we prove: For all $n$, if $\langle \alpha_0, \ldots, \alpha_n \rangle$ is a deduction from $\Sigma$ of $\alpha_n$ then $\Sigma \models \alpha_n$.

- (Base case) If $n = 0$ then the hypothesis is that $\langle \alpha_0 \rangle$ is a deduction from $\Sigma$ of $\alpha_0$. By definition of a deduction, this means that either
  
  $- \alpha_0$ is a tautology. In this case, $\Sigma \models \alpha_0$ because every truth assignment satisfies $\alpha_0$.
  
  Otherwise,
  
  $- \alpha_0 \in \Sigma$. In this case, $\Sigma \models \alpha_0$ because every truth assignment that satisfies each wff in $\Sigma$ must (by definition) satisfy $\alpha_0$.

- (Inductive step) The inductive hypothesis is that for all $m \leq n$, if $\langle \alpha_0, \ldots, \alpha_m \rangle$ is a deduction from $\Sigma$ of $\alpha_m$ then $\Sigma \models \alpha_m$. Suppose $\langle \alpha_0, \ldots, \alpha_{n+1} \rangle$ is a deduction from $\Sigma$ of $\alpha_{n+1}$. Then, $\alpha_{n+1}$ is either a tautology, in $\Sigma$, or obtainable from earlier $\alpha_i, \alpha_j$ by modus ponens. In the first two cases $\Sigma \models \alpha_{n+1}$ as in the base case. So, let $i, j < n$ be such that $\alpha_i = \alpha_j \rightarrow \alpha_{n+1}$. Notice that $\langle \alpha_0, \ldots, \alpha_i \rangle, \langle \alpha_0, \ldots, \alpha_j \rangle$ are each deductions and have length less than or equal to $n$. By the induction hypothesis, $\Sigma \models \alpha_i$ and $\Sigma \models \alpha_j$. That is,

$$\Sigma \models \alpha_j \rightarrow \alpha_{n+1} \quad \text{and} \quad \Sigma \models \alpha_j.$$

To prove that $\Sigma \models \alpha_{n+1}$, let $v$ be a truth assignment that satisfies all wffs in $\Sigma$. Then,

$$\bar{v}(\alpha_j \rightarrow \alpha_{n+1}) = T \quad \text{and} \quad \bar{v}(\alpha_j) = T.$$

This implies that $\bar{v}(\alpha_{n+1}) = T$ and we have shown that $\Sigma \models \alpha_{n+1}$. $\square$
Definition (p. 59). A set $\Sigma$ of wffs is **satisfiable** if there is a truth assignment that satisfies every member of $\Sigma$.

**Definition.** A set $\Sigma$ of wffs is **consistent** if there is no wff $\varphi$ such that $\Sigma \vdash \varphi$ and $\Sigma \vdash \neg \varphi$.

**Fact** ((a) on p. 60). (a) $\Sigma \vdash \tau$ iff $\Sigma \cup \{\neg \tau\}$ is unsatisfiable.

(b) $\Sigma \vdash \tau$ iff $\Sigma \cup \{\neg \tau\}$ is inconsistent.

**Proof.** (a) Suppose $\Sigma \vdash \tau$: every assignment that satisfies $\Sigma$ satisfies $\tau$. Thus, every assignment that satisfies $\Sigma$ does **not** satisfy $\neg \tau$. Hence, no assignment can satisfy $\Sigma \cup \{\neg \tau\}$.

Conversely, suppose $\Sigma \cup \{\neg \tau\}$ is unsatisfiable and let $v$ be a truth assignment that satisfies $\Sigma$. By assumption, $\bar{v}(\neg \tau) \neq T$ so $\bar{v}(\tau) = T$. Thus, $\Sigma \vdash \tau$.

(b) Suppose $\Sigma \vdash \tau$. Since adding hypotheses doesn’t impact provability, $\Sigma \cup \{\neg \tau\} \vdash \tau$. But also (by definition), $\Sigma \cup \{\neg \tau\} \vdash \neg \tau$. Therefore, $\Sigma \cup \{\neg \tau\}$ is not consistent.

For the converse, we will use the following lemma: for a set of wffs $\Gamma$ and wffs $\varphi, \psi$, if $\Gamma \cup \{\varphi\} \vdash \psi$ then $\Gamma \vdash \varphi \rightarrow \psi$. We prove this by the length of the deduction witnessing $\Gamma \cup \{\varphi\} \vdash \psi$. In the base case, the deduction is simply $\langle \psi \rangle$ and $\psi \in \Gamma \cup \{\varphi\}$ or $\vdash \psi$. If $\varphi = \psi$ then $\vdash \varphi \rightarrow \psi$ so $\Gamma$ proves it. Otherwise, $\psi \in \Gamma$ or $\vdash \psi$ and (either way) $\langle \psi, (\psi \rightarrow (\varphi \rightarrow \psi)), \varphi \rightarrow \psi \rangle$ is a deduction from $\Gamma$ since $\vdash (\psi \rightarrow (\varphi \rightarrow \psi))$. The induction hypothesis is that $\psi$ is obtainable from $\alpha, \beta$ by modus ponens and $\Gamma \vdash \varphi \rightarrow \alpha$; $\Gamma \vdash \varphi \rightarrow \beta$. I.e., $\beta = \alpha \rightarrow \psi$. Notice that $\vdash (\varphi \rightarrow \alpha) \rightarrow ((\varphi \rightarrow (\alpha \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi))$.

Concatenating the deductions from the inductive hypothesis with this tautology and applying modus ponens twice gives $\Gamma \vdash \varphi \rightarrow \psi$, as required. Back to proof: suppose $\Sigma \cup \{\neg \tau\}$ is inconsistent. Then there is some wff $\gamma$ such that $\Sigma \cup \{\neg \tau\} \vdash \gamma$ and $\Sigma \cup \{\neg \tau\} \vdash \neg \gamma$. By Lemma, $\Sigma \vdash (\neg \tau \rightarrow \gamma)$ and $\Sigma \vdash (\neg \tau \rightarrow \neg \gamma)$. Truth tables confirm that $\vdash (\neg \tau \rightarrow \gamma) \rightarrow ((\neg \tau \rightarrow \neg \gamma) \rightarrow \tau)$ (the intuition is that if $\neg \tau$ implies both a wff and its negation then it can’t be true). Hence concatenating the deductions for $\Sigma \vdash (\neg \tau \rightarrow \gamma)$ and $\Sigma \vdash (\neg \tau \rightarrow \neg \gamma)$, with this tautology and applying modus ponens twice gives a deduction witnessing $\Sigma \vdash \tau$. 

**Theorem.** Soundness is equivalent to “every satisfiable set of formulas is consistent” and completeness is equivalent to “every consistent set of formulas is satisfiable.”

**Proof.** Homework 3.

**Proof of completeness.** We will prove the equivalent statement: every consistent set of formulas is satisfiable. Let $\Sigma$ be a consistent set of wffs. We will extend $\Sigma$ to a **negation-complete** and consistent set of wffs, $\Theta$ (that is, $\Theta$ is consistent and for every $\varphi$ either $\varphi \in \Theta$ or $\neg \varphi \in \Theta$). Then, we define a truth assignment that satisfies all wffs in $\Theta$ (and hence also all in $\Sigma$). Let $\alpha_0, \alpha_1, \ldots$ be an enumeration of all wffs. Define

\[
\Theta_0 = \Sigma \\
\Theta_{n+1} = \begin{cases} \Theta_n \cup \{\alpha_n\} & \text{if } \Theta_n \cup \{\alpha_n\} \text{ is consistent} \\
\Theta_n \cup \{\neg \alpha_n\} & \text{otherwise} \\
\end{cases} \\
\Theta = \bigcup_n \Theta_n
\]

It is immediate that $\Theta$ is negation-complete.
Why is $\Theta$ consistent? First note that each $\Theta_n$ is consistent:

- $\Theta_0 = \Sigma$ is assumed consistent.
- Consistency is preserved at each step: Suppose $\Theta_n$ is consistent. If $\Theta_n \cup \{\alpha_n\}$ is consistent then $\Theta_{n+1} = \Theta_n \cup \{\alpha_n\}$ and we’re done. Otherwise, we defined $\Theta_{n+1} = \Theta_n \cup \{-\alpha_n\}$. By Fact (b), in this case, $\Theta_n \vdash -\alpha_n$. Thus, for any $\varphi$ that $\Theta_{n+1} \vdash \varphi$, it is also the case that $\Theta_n \vdash \varphi$. Hence, since $\Theta_n$ is consistent, so is $\Theta_{n+1}$.

Suppose, for a contradiction, that $\Theta$ is not consistent. Then there is some $\varphi$ such that $\Theta \vdash \varphi$ and $\Theta \vdash -\varphi$. Each of these is witnessed by a finite derivation so the union of these two derivations is finite as well. Let $k$ be smallest such that all these (finitely many) formulas appear in $\Theta_k$. Then $\Theta_k \vdash \varphi$ and $\Theta_k \vdash -\varphi$. But, this contradicts the consistency of $\Theta_k$.

By consistency and negation-completeness, for each wff $\alpha$, exactly one of $\Theta \vdash \alpha$ and $\Theta \vdash -\alpha$ holds. Why? At most one holds by consistency. And, if $\Theta \not\vdash \alpha$ then by Fact (b), $\Theta \cup \{-\alpha\}$ is consistent. Since $-\alpha$ appears on list of wffs, say $\alpha_k = -\alpha$, then $\Theta_k \cup \{-\alpha\}$ is consistent and hence $-\alpha \in \Theta_k \subseteq \Theta$. In this case, $\Theta \vdash -\alpha$. Similarly, if $\Theta \not\vdash -\alpha$ then $\Theta \vdash \alpha$.

The above argument holds, in particular, for sentence symbols so the following definition is total and non-contradictory: let $v$ be the truth assignment

$$v(A_i) = \begin{cases} T & \text{if } \Theta \vdash A_i \\ F & \text{if } \Theta \not\vdash A_i \end{cases}$$

We will prove by induction that for each wff $\varphi$, $v(\varphi) = T$ if and only if $\Theta \vdash \varphi$. This is an induction on the structure of formulas, and since we have proved that every wff is tautologically equivalent to one built up from sentence connectives using only $-\,$ and $\wedge$ (because $\{-,\wedge\}$ is a complete sets of connectives), it is sufficient to consider just these wffs.

- (Base case) Consider $\varphi = A_i$. Then $v(A_i) = v(A_i) = T$ if and only if $\Theta \vdash A_i$ by definition.
- (Inductive step) Let $\alpha, \beta$ be wffs such that $v(\alpha) = T$ if and only if $\Theta \vdash \alpha$ and $v(\beta) = T$ if and only if $\Theta \vdash \beta$.
  - Let $\varphi = -\alpha$. Then by consistency and negation completeness,
    $$\Theta \vdash \varphi \iff \Theta \not\vdash \alpha \iff v(\alpha) = F \iff v(\varphi) = T.$$
  - Let $\varphi = \alpha \wedge \beta$. Since $\vdash \varphi \rightarrow \alpha$, $\vdash \varphi \rightarrow \beta$ if $\Theta \vdash \varphi$ then $\Theta \vdash \alpha$ and $\Theta \vdash \beta$. Conversely, if $\Theta \vdash \alpha$ and $\Theta \vdash \beta$ then $\Theta \vdash \varphi$ because $\vdash \alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta))$.
    $$\Theta \vdash \varphi \iff \begin{cases} \Theta \vdash \alpha & \text{iff } v(\alpha) = T \\ \Theta \vdash \beta & \text{iff } v(\beta) = T \end{cases} \iff v(\varphi) = T.$$

For each $\alpha \in \Sigma$, $\Sigma \vdash \alpha$ and hence (since $\Sigma \subseteq \Theta$) $\Theta \vdash \alpha$. Therefore, $v(\alpha) = T$ and $v$ satisfies $\Sigma$.

**Theorem** (Compactness of sentential logic, p. 24 and 59). Let $\Sigma$ be an infinite set of wffs such that every finite subset is satisfiable. Then $\Sigma$ itself is satisfiable.

**Proof.** One direction is immediate: if $\Sigma$ satisfiable then there is $v$ which satisfies every member of $\Sigma$ and therefore which satisfies every member of every finite subset.
For the converse, we can do a construction analogous to the proof of completeness. For details: see proof in book (page 59-60 and 1.7#1,2) and practice questions for exam 1.

An alternate proof assumes soundness and completeness. Suppose every consistent set of formulas is satisfiable. Let $\Sigma$ be an infinite set of wffs such that every finite subset is satisfiable. By soundness, this implies every finite subset is consistent. Suppose (for a contradiction) that $\Sigma$ is not satisfiable and hence (by completeness) not consistent. Then there is $\varphi$ such that $\Sigma \vdash \varphi$ and $\Sigma \vdash \neg \varphi$. Let $\Sigma_\varphi$ be the set of all formulas that appear in either of the deductions witnessing these two facts. Then $\Sigma_\varphi \subseteq \Sigma$ is finite (because deductions are finite) and $\Sigma_\varphi \vdash \varphi$ and $\Sigma_\varphi \vdash \neg \varphi$. That is, $\Sigma_\varphi$ is a finite inconsistent subset, a contradiction.

$\square$

**Corollary** (17A on p. 60). Compactness is equivalent to “If $\Sigma \models \tau$ then there is a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \tau$”.

**Proof.** Suppose every finitely satisfiable $\Sigma$ is satisfiable and $\Sigma \models \tau$. Then $\Sigma \cup \{\neg \tau\}$ is unsatisfiable (by Fact (a)) and therefore there is a finite subset $\Sigma_0 \subseteq \Sigma \cup \{\neg \tau\}$ that is unsatisfiable. Then $\Sigma_0 \cup \{\neg \tau\}$ (which may equal $\Sigma_0$) is unsatisfiable and hence (by Fact (a)) $\Sigma_0 \models \tau$.

Conversely, suppose that if $\Sigma \models \tau$ there is a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \tau$. Towards a contradiction, let $\Sigma$ be an infinite set of wffs such that each finite subset is satisfiable but $\Sigma$ is not satisfiable. Since $\Sigma$ is nonempty, let $\varphi$ be some wff in $\Sigma$ and denote $\Sigma \setminus \{\varphi\}$ by $\Sigma'$. Then $\Sigma' \cup \{\varphi\}$ is not satisfiable so by Fact (a), $\Sigma' \models \neg \varphi$. By the assumption, there is a finite $\Sigma_0 \subseteq \Sigma' \subseteq \Sigma$ such that $\Sigma_0 \models \neg \varphi$. Applying Fact (a) again, we conclude that $\Sigma_0 \cup \{\varphi\}$ is unsatisfiable. But, this is a finite subset of $\Sigma$, contradicting our assumption that $\Sigma$ is finitely satisfiable.

$\square$