Prove that $D_4/Z(D_4)$ is isomorphic to $Z_2 \oplus Z_2$. 
We start by computing $Z(D_4)$, the center of $D_4$:

$$Z(D_4) = \{e, R_{180}\}$$

(for each other element $x$ of $D_4$, there is at least one $y \in D_4$ such that $xy \neq yx$). By Lagrange's Theorem,

$$|D_4/Z(D_4)| = \frac{|D_4|}{|Z(D_4)|} = \frac{8}{2} = 4.$$  

Listing the cosets of $Z(D_4)$ in $D_4$, we see that the elements of the factor group $D_4/Z(D_4)$ are

$$Z(D_4) = \{e, R_{180}\}, \quad R_{90}Z(D_4) = \{R_{90}, R_{270}\},$$

$$FZ(D_4) = \{F, FR_{180}\}, \quad FR_{90}Z(D_4) = \{FR_{90}, FR_{270}\}.$$  

By the characterization of groups of order 4 (Example 3 on p. 163), either $D_4/Z(D_4)$ is cyclic or it is isomorphic to $Z_2 \oplus Z_2$. Therefore, it is sufficient to compute the order of each element of the factor group and prove that it is strictly less than 4. For brevity, let $H = Z(D_4)$.

$$(H)^2 = (eH)(eH) = (e \circ e)H = eH = H.$$  

$$(R_{90}H)^2 = (R_{90}H)(R_{90}H) = (R_{90} \circ R_{90})H = R_{180}H = H.$$  

$$(FH)^2 = (FH)(FH) = (F \circ F)H = eH = H.$$  

$$(FR_{90}H)^2 = (FR_{90}H)(FR_{90}H) = (FR_{90} \circ FR_{90})H = (F^2R_{90}R_{270})H = eH = H.$$  

Thus, since $H$ is the identity element of the factor group, $|X| \leq 2$ for each $X \in D_4/H$ and our proof is complete.