Assigned reading: Chapters 7, (parts of 31), 5 of Gallian.

Recommended practice questions:
Chapter 7 of Gallian, exercises 42, 44
Chapter 31 of Gallian, exercises 5, 10
Chapter 5 of Gallian, exercises 1, 2, 3, 12, 73
Chapter 6 of Gallian, exercises 19, 47

Assigned questions to hand in:

(1) (Gallian Chapter 0 # 52) The ISBN-10 0−669−03925−4 is the result of a transposition of two adjacent digits not involving the first or last digit. Determine the correct ISBN-10. Justify your answer.

Solution: We can confirm that the given string of digits is not a valid ISBN-10 code:

\[ 11 - (10 \cdot 0 + 9 \cdot 6 + 8 \cdot 6 + 7 \cdot 9 + 6 \cdot 0 + 5 \cdot 3 + 4 \cdot 9 + 3 \cdot 2 + 2 \cdot 5 \mod 11) \]
\[ = 11 - (0 + 54 + 48 + 63 + 0 + 15 + 36 + 6 + 10 \mod 11) \]
\[ = 11 - (0 + (-1) + 4 + (-3) + 0 + 4 + 3 + 6 + (-1) \mod 11) \]
\[ = 11 - 1 = 10 \neq 4. \]

Since the check digit as computed from the given string is greater than the correct digit, the transposition must have put a larger number to the right of where it is supposed to be. Therefore, the possible transpositions are:

\[ 6 \leftrightarrow 9, \quad 0 \leftrightarrow 3, \quad 3 \leftrightarrow 9, \quad 2 \leftrightarrow 5. \]

Try: suppose correct values for the first nine digits are 0−696−03925. Then the check digit would be

\[ 11 - (10 \cdot 0 + 9 \cdot 6 + 8 \cdot 9 + 7 \cdot 6 + 6 \cdot 0 + 5 \cdot 3 + 4 \cdot 9 + 3 \cdot 2 + 2 \cdot 5 \mod 11) \]
\[ = 11 - (0 + 54 + 72 + 42 + 0 + 15 + 36 + 6 + 10 \mod 11) \]
\[ = 11 - (0 + (-1) + (-5) + (-2) + 0 + 4 + 3 + 6 + (-1) \mod 11) \]
\[ = 11 - 4 = 7 \neq 4. \]
Try: suppose correct values for the first nine digits are 0 – 669 – 30925. Then the check digit would be

\[
11 - (10 \cdot 0 + 9 \cdot 6 + 8 \cdot 6 + 7 \cdot 9 + 6 \cdot 3 + 5 \cdot 0 + 4 \cdot 9 + 3 \cdot 2 + 2 \cdot 5 \mod 11)
\]
\[
= 11 - (0 + 54 + 48 + 63 + 0 + 36 + 6 + 10 \mod 11)
\]
\[
= 11 - (0 + (-1) + 4 + (-3) + 7 + 0 + 3 + 6 + (-1) \mod 11)
\]
\[
= 11 - 4 = 7 \neq 4.
\]

Try: suppose correct values for the first nine digits are 0 – 669 – 09325. Then the check digit would be

\[
11 - (10 \cdot 0 + 9 \cdot 6 + 8 \cdot 6 + 7 \cdot 9 + 6 \cdot 0 + 5 \cdot 9 + 4 \cdot 3 + 3 \cdot 2 + 2 \cdot 5 \mod 11)
\]
\[
= 11 - (0 + 54 + 48 + 63 + 0 + 45 + 12 + 6 + 10 \mod 11)
\]
\[
= 11 - (0 + (-1) + 4 + (-3) + 0 + 1 + 1 + 6 + (-1) \mod 11)
\]
\[
= 11 - 4 = 7.
\]

Thus, the correct ISBN-10 is 0 – 669 – 09325 – 4.

(2) (Gallian Chapter 5 # 6) What is the order of each of the following permutations? Justify your answers.

(a) \[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
2 & 1 & 5 & 4 & 6 & 3
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
7 & 6 & 1 & 2 & 3 & 4 & 5
\end{pmatrix}
\]

Solution: We write each permutation in disjoint cycle form and then the order of the permutation is the lcm of the lengths of the cycles.

(a) \((1,2)(3,5,6)(4)\)

The cycles have length 2, 3, 1 so the lcm is 6.

(b) \((1,7,5,3)(2,6,4)\)

The cycles have length 4, 3 so the lcm is 12.

(3) (Gallian Chapter 5 # 9) What are the possible orders for the elements of \(S_6\) and \(A_6\)? What about \(A_7\)? Justify your answers.

Solution: The order of an element in \(S_6\) is the lcm of lengths of cycles of elements in \(S_6\). The possible cycle lengths are 1, 2, 3, 4, 5, 6 which can be combined as:

- One cycle of length 6,
- One cycle of length 5 and one of length 1,
- One cycle of length 4 and one of length 2,
- One cycle of length 4 and two of length 1,
- Two cycles of length 3,
- One cycle of length 3, one cycle of length 2, and one cycle of length 1,
- One cycle of length 3 and three cycles of length 1,
- Three cycles of length 2,
- Two cycles of length 2 and two cycles of length 1,
- One cycle of length 2 and four cycles of length 1,
- Six cycles of length 1.
The resulting orders of elements are, respectively, 6, 5, 4, 4, 3, 6, 3, 2, 2, 1. Thus, the possible orders of elements are 1, 2, 3, 4, 5, 6.

The elements of $A_6$ are even permutations, those that can be written as a product of an even number of length-2 cycles. Recall that a cycle of odd length is an even permutation and a cycle of even length is an odd permutation. Thus, the list of possible cycle combinations in $A_6$ is

- One cycle of length 6,
- One cycle of length 5 and one of length 1,
- One cycle of length 4 and one of length 2,
- One cycle of length 3 and two of length 1,
- Two cycles of length 3,
- One cycle of length 3, one cycle of length 2, and one cycle of length 1.
- One cycle of length 3 and three cycles of length 1,
- Three cycles of length 2,
- Two cycles of length 2 and two cycles of length 1,
- One cycle of length 2 and four cycles of length 1,
- Six cycles of length 1.

The resulting orders of elements are, respectively, 5, 4, 3, 3, 2, 1.

For $A_7$, we have

- $(1, 2, 3, 4, 5, 6, 7)$ is a cycle of length 7 so is even and in $A_7$. It has order 7.
- $(1, 2, 3)(4, 5)(6, 7) = (1, 2)(1, 3)(4, 5)(6, 7)$ so is a product of an even number of transpositions. It has order $lcm(3, 2, 2) = 6$.
- $(1, 2, 3, 4, 5)(6, 7)$ is a cycle of length 5 so is in $A_7$. It has order 5.
- $(1, 2, 3, 4)(5, 6)(7) = (1, 2)(1, 3)(1, 4)(5, 6)$ so is a product of an even number of transpositions. It has order $lcm(4, 2, 1) = 4$.
- $(1, 2, 3)(4, 5, 6)(7) = (1, 2)(1, 3)(4, 5)(4, 6)$ so is a product of an even number of transpositions. It has order $lcm(3, 3, 1) = 3$.
- $(1, 2)(3, 4)(5, 6)(7) = (1, 2)(3, 4)$ and has order 2.
- $e = (1, 2)(2, 1) = (1)(2)(3)(4)(5)(6)(7)$ has order 1.

Thus, the possible orders of elements are 1, 2, 3, 4, 5, 6, 7.

(4) (Gallian Chapter 5 # 14) Find eight elements in $S_6$ that commute with $(12)(34)(56)$. Do they form a subgroup of $S_6$?

**Solution:** Consider the elements

$\{ (5, 6), (1, 2), (3, 4), (5, 6)(1, 2), (5, 6)(3, 4), (1, 2)(3, 4), (5, 6)(1, 2)(3, 4), e = (1, 2)(2, 1) \}$.

In array form, these are

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 2 & 3 & 4 & 6 & 5
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
2 & 1 & 3 & 4 & 5 & 6
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 2 & 3 & 4 & 5 & 6
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
2 & 1 & 3 & 4 & 6 & 5
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
2 & 1 & 3 & 4 & 5 & 6
\end{pmatrix}
$$

Each commutes with $(12)(34)(56)$ because disjoint cycles commute and inverse pairs commute (and a transposition is its own inverse).

$$((12)(34)(56))(56) = (12)(34) = (56)((12)(34)(56))$$

The Cayley table is
(5) (Gallian Chapter 5 # 74) Let \( H = \{ \alpha^2 : \alpha \in S_6 \} \). Prove that \( H \neq A_6 \).

**Solution:** It is enough to find an even permutation in \( S_6 \) that is not the square of any other permutation. In other words, we will show that \( A_6 \nsubseteq H \) and hence \( A_6 \neq H \). Consider the permutation \( f = (1, 2, 3, 4)(5, 6) \). Suppose, towards a contradiction that \( f \in H \). Let \( \alpha \in S_6 \) be such that \( \alpha^2 = f \). Consider the possible values of \( \alpha(1) \).

- If \( \alpha(1) = 1 \) then \( \alpha^2(1) = 1 \neq f(1) \), a contradiction.
- If \( \alpha(1) = 2 \) then to have \( \alpha^2 = f \), it must be the case that \( \alpha(2) = 2 \), but then \( \alpha \) is not one-to-one, a contradiction.
- If \( \alpha(1) = 3 \) then to have \( \alpha^2 = f \), it must be the case that
  \[
  \alpha(3) = 2, \quad \alpha(2) = 4, \quad \alpha(4) = 3
  \]
  but then \( \alpha \) is not one-to-one, a contradiction.
- If \( \alpha(1) = 4 \) then to have \( \alpha^2 = f \), it must be the case that
  \[
  \alpha(4) = 2, \quad \alpha(2) = 1, \quad \alpha(1) = 3,
  \]
  but then \( \alpha \) is not well-defined, a contradiction.
- If \( \alpha(1) = 5 \) then to have \( \alpha^2 = f \), it must be the case that
  \[
  \alpha(5) = 2, \quad \alpha(2) = 6, \quad \alpha(6) = 3, \quad \alpha(3) = 5,
  \]
  but then \( \alpha \) is not one-to-one, a contradiction.
- Finally, if \( \alpha(1) = 6 \), then to have \( \alpha^2 = f \), it must be the case that
  \[
  \alpha(6) = 2, \quad \alpha(2) = 5, \quad \alpha(5) = 3, \quad \alpha(3) = 6,
  \]
  but then \( \alpha \) is not one-to-one, a contradiction.

Since all possible values of \( \alpha(1) \) lead to a contradiction, \( f \) cannot be written as the square of any permutation in \( S_6 \) and \( f \notin H \).

**Note:** it is possible to show that \( H \nsubseteq A_6 \); that is, that every square is an even permutation.

(6) (Gallian Supplementary 5-8 # 62) Let \( H = \{ \alpha \in S_n : \alpha \text{ maps the set } \{1, 2\} \text{ to itself} \} \). Prove that \( C((1, 2)) = H \).

**Solution:** We prove subset containment in two directions.

- Suppose \( \alpha \in H \). That is \( \alpha(1) \in \{1, 2\} \) and \( \alpha(2) \in \{1, 2\} \). We want to show that \( \alpha(1, 2) = (1, 2)\alpha \).
If $\alpha/D_4^1/D_5^1/AG_1$ then (since $\alpha$ is a permutation) $\alpha/D_4^2/D_5^2/AG_2$.

For $x = 1$

\[(\alpha(1, 2))(x) = \alpha(2) = 2 \quad (1, 2)\alpha(x) = (1, 2)\alpha(1) = 2,\]

for $x = 2$,

\[(\alpha(1, 2))(x) = \alpha(1) = 1 \quad (1, 2)\alpha(x) = (1, 2)\alpha(2) = 1,\]

and for $x \in \{3, \ldots, n\}$, $\alpha(x) \in \{3, \ldots, n\}$ as well and,

\[(\alpha(1, 2))(x) = \alpha(x) \quad (1, 2)\alpha(x) = (1, 2)\alpha(x) = \alpha(x).\]

If $\alpha(1) = 2$. Then for $x = 1$

\[(\alpha(1, 2))(x) = \alpha(2) = 1 \quad (1, 2)\alpha(x) = (1, 2)\alpha(1) = 1,\]

for $x = 2$,

\[(\alpha(1, 2))(x) = \alpha(1) = 2 \quad (1, 2)\alpha(x) = (1, 2)\alpha(2) = 2,\]

and for $x \in \{3, \ldots, n\}$,

\[(\alpha(1, 2))(x) = \alpha(x) \quad (1, 2)\alpha(x) = (1, 2)\alpha(x) = \alpha(x).\]

Suppose $\alpha \in C(1, 2)$. We want to show that $\alpha(1) \in \{1, 2\}$ and $\alpha(2) \in \{1, 2\}$. For a contradiction, assume that $\alpha(b) = c \notin \{1, 2\}$ for some $b \in \{1, 2\}$. By definition of the centralizer,

\[\alpha(1, 2) = (1, 2)\alpha.\]

Consider the image of $b$:

\[LHS : (\alpha(1, 2))(b) = \alpha(\bar{b})\]

\[RHS : ((1, 2)\alpha)(b) = (1, 2)\alpha(b) = (1, 2)(c) = c = \alpha(b).\]

because $c$ is fixed by the permutation $(1, 2)$, and where we define $\bar{b} = 1$ if $b = 2$ and $\bar{b} = 2$ if $b = 1$. Since $LHS = RHS$, we see that $\alpha(\bar{b}) = \alpha(b)$, but this contradicts $\alpha$ being one-to-one.