Math 20D Midterm Exam
Practice Problems

1. A tank contains \( G \) gallons of fresh water. A solution with a concentration of \( c \) lb of salt per gallon is pumped into the tank at a rate of \( r \) gallons per minute. The well-mixed solution drains from the tank at the same rate. Let \( Q(t) \) be the amount of salt in the tank after \( t \) minutes, measured in lbs.

(a) Write an initial value problem describing the amount of salt in the tank at time \( t \).

(b) Solve the initial value problem in part (a) for \( Q(t) \).

(c) What are the equilibrium state(s) of the system? Are these stable, unstable, or semistable?

2. Find the general solution to each of the following differential equations:

(a) \( y' + \frac{2}{t} y = \frac{\cos t}{t^2} \)

(b) \( \frac{du}{dx} = \frac{x^2}{y(1 + x^3)} \)

(c) \( \frac{du}{dx} = \frac{4y - 3x}{2x - y} \)

(d) \( y'' + 4y' - 21y = 0 \)

(e) \( y'' - 8y' + 16y = 0 \)

(f) \( y'' + 10y' + 34y = 0 \)

3. Determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.

(a) \( y' + (\tan t)y = \frac{1}{t}, \quad y(\pi) = 2 \)

(b) \( (4 - t^2)y' + y = \frac{1}{t^2}, \quad y(3) = -1 \)

4. Suppose a 25-year-old opens a retirement savings account earning an annual interest rate \( r \) with an initial investment of $1000. They then deposit $\( m \) per year. Assume that compounding and deposits occur continuously. Let \( S(t) \) be the amount of money in the account after \( t \) years.

(a) Write an initial value problem describing the amount of money in the account at time \( t \).
(b) Solve the initial value problem in part (a).

(c) If the interest rate is 5% and the account holder invests $1200 per year, how long will it take before the account has $1,000,000?

5. Consider the autonomous equation

\[ \frac{dy}{dt} = 3y^2(y^2 - 9) \]

(a) Determine the equilibrium solution(s).

(b) Draw the phase line and classify the equilibrium solution(s) as stable, unstable, or semistable.

(c) Sketch several graphs of solutions in the \( ty \)-plane.

6. Show that \( \mu(x, y) = y \cos x \) is an integrating factor that makes the following differential equation exact:

\[ -y \tan x + y^{-1} + 2(1 - \sec x) \frac{dy}{dx} = 0. \]

Use the integrating factor to solve the differential equation.

7. Consider the differential equation

\[ xy'' - y' - 4x^3y = 0. \]

(a) Show that \( y_1(x) = e^{-x^2} \) is a solution.

(b) Use reduction of order to find a second solution of the form \( y_2(x) = v(x)e^{-x^2} \).

8. Newton’s law of cooling states that the temperature of an object changes at a rate proportional to the difference of its temperature and the temperature of its environment. Suppose there is a cup of tea sitting on the kitchen table. We measure its temperature to be 195 degrees Fahrenheit. We leave the tea for one minute and then measure its temperature again, getting a second temperature of 185 degrees Fahrenheit.

(a) Assuming the air temperature in the kitchen is 70 degrees Fahrenheit, find an equation \( T(t) \) giving the temperature of the tea after \( t \) minutes.

(b) Use the above equation to determine the time at which the tea will be 150 degrees Fahrenheit.