Math 171B Practice Final Exam

1. Find all the local minimizers of
\[
\begin{align*}
\min & \quad x_1^2 - 2x_2^2 + 4x_1x_2 \\
\text{s.t.} & \quad x_1^2 + x_2^2 - 1 = 0
\end{align*}
\]
and their Lagrange multiplier. Which one of them is a global minimizer?

2. Find all the local minimizer of the optimization problem
\[
\begin{align*}
\min & \quad -x_1^2x_2^2 \\
\text{s.t.} & \quad 1 - x_1^2 \geq 0, \\
& \quad 1 - x_2^2 \geq 0,
\end{align*}
\]
and their associated Lagrange multipliers, which one of them is a global minimizer?

3. Suppose a sequence \( \{x_k\} \) is generated by the iteration formula,
\[
x_{k+1} = \frac{1}{3}(2x_k + \frac{1}{x_k^2})
\]
with the starting point \( x_0 > 0 \). Suppose \( \{x_k\} \) converges, what is the limit and convergence order? Justify your answer.

4. Let \( H \in \mathbb{R}^{n \times n} \) be a symmetric matrix. Suppose
\[
a^T x + \frac{1}{2} x^T H x \geq 0 \quad \text{for all} \quad x \in \mathbb{R}^n.
\]
Show that \( a = 0 \) and \( H \) is positive semidefinite.

5. Let \( f(x) : \mathbb{R}^n \to \mathbb{R} \) be a scalar valued twice differentiable function, and \( u \in \mathbb{R}^n \) be such that \( \nabla f(u) \neq 0 \). Suppose \( p^* \) is a minimizer of the quadratic function
\[
\varphi(p) := \nabla f(u)^T p + \frac{1}{2} p^T B p
\]
where \( B \succ 0 \). Show that \( p^* \) is a descent direction \( f(x) \) at \( u \).

6. Let \( A \in \mathbb{R}^{n \times n} \) be a nonsingular matrix, and \( f(x) : \mathbb{R}^m \to \mathbb{R} \) be a scalar valued function. Suppose \( \nabla f(0) = 0 \) and \( \nabla^2 f(0) \succ 0 \). Define the new function
\[
g(x) := f(Ax).
\]
Show that the origin is a strict local minimizer of \( g(x) \).

7. Consider the quadratic optimization
\[
(QP) : \min \quad x^T Ax \\
\text{s.t.} \quad 1 - x^T x \geq 0,
\]
where \( A \in \mathbb{R}^{n \times n} \) is symmetric. Let \( x^* \) be a local minimizer and \( \lambda^* \) be its Lagrange multiplier. If \( \lambda_{\min}(A) + \lambda^* \geq 0 \), show that \( x^* \) is a global minimizer of this QP.
1. Find all the local minimizers of
\[\min x_1^2 - 2x_2^2 + 4x_1x_2\]
s.t. \(x_1^2 + x_2^2 - 1 = 0\)
and their Lagrange multiplier. Which one of them is a global minimizer?

Solution: By optimality condition:
\[\nabla f(x) = \lambda \nabla c(x).\]

So
\[\begin{bmatrix} 2x_1 + 4x_2 \\ -4x_2 + 4x_1 \end{bmatrix} = \lambda \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}.
\]

With constraints \(x_1^2 + x_2^2 = 1\), we have \(\lambda = 2\) or \(\lambda = -3\).

If \(\lambda = 2\), we have \(x_1 = 2x_2 = \pm 2\sqrt{\frac{1}{5}}\).

If \(\lambda = -3\), \(x_2 = -2x_1 = \pm 2\sqrt{\frac{1}{5}}\).

Check the second order condition:
\[\nabla^2 f(x) - \lambda \nabla^2 c(x) = \begin{bmatrix} 2 & 4 \\ 4 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix}.
\]

So for any \(p \in \text{Null}(J_c(x^*))\), second order condition is true. We find 2 minimizers with \(\lambda = -3\),

\[x_1^* = \sqrt{\frac{1}{5}}, x_2^* = -2\sqrt{\frac{1}{5}}\]

another minimizer \(x_1^* = -\sqrt{\frac{1}{5}}, x_2^* = 2\sqrt{\frac{1}{5}}\).

Global minimum \(f(x^*) = -\frac{1}{3}\).

2. Find all the local minimizer of the optimization problem
\[\min -x_1^2 x_2^2\]
s.t. \(1 - x_1^2 \geq 0\)
\(1 - x_2^2 \geq 0\)
and their associated Lagrange multipliers, which one of them is a global minimizer?

Solution: (1) If active constraint \(A = \{1\}\), then \(c_1(x) = 0, x_1 = \pm 1\). First order optimality condition \(\nabla f(x) = \lambda \nabla c_1(x)\), then
\[\begin{bmatrix} -2x_1^2 \\ -2x_2^2 \end{bmatrix} = \lambda \begin{bmatrix} -2x_1 \\ 0 \end{bmatrix},\]
we have \(\lambda = 0\) and \(x_2 = 0\), which satisfy the second constraint. Check second order condition:
\[\nabla^2 f(x) - \lambda \nabla^2 c_1(x) = \begin{bmatrix} -2x_2^2 & -4x_1x_2 \\ -4x_1x_2 & -2x_1^2 \end{bmatrix}.
\]
For \( p \in \text{Null}(\nabla c_1(x)) \), we have \( 2x_1p_1 = 0 \), and \( x_1 \neq 0 \), so \( p_1 = 0, p_2 \in \mathbb{R} \). Check for \( p \in \text{Null}(\nabla c_1(x)) \), second order condition is not true.

Similarly, if active constraint \( A = \{2\} \), we still find second order condition is not true.

Then active constraints \( A = \{1, 2\} \), in this case, there are only 4 feasible points \((1, 1), (1, -1), (-1, 1), (-1, -1)\).

First order condition

\[ \nabla f(x) = \lambda_1 \nabla c_1(x) + \lambda_2 \nabla c_2(x), \]

i.e.

\[ \begin{bmatrix} x_2^2 \\ x_1^2 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \]

For \( x = (1, 1), (1, -1), (-1, 1), (-1, -1) \), \( \lambda = (1, 1) \geq 0 \); and null space of \( \nabla c_1(x), \nabla c_2(x) \) only has one point \((0, 0)\), so second order condition is also true. The four points all local minimizers, and they have the same objective value, so they are all global minimizers. \( f(x^*) = -1 \).

3. Suppose a sequence \( \{x_k\} \) is generated by the iteration formula,

\[ x_{k+1} = \frac{1}{3}(2x_k + \frac{1}{x_k^2}) \]

with the starting point \( x_0 > 0 \). Suppose \( \{x_k\} \) converges, what is the limit and convergence order? Justify your answer.

Solution: Let \( x^* \) be the limit point, and since the sequence converges, so \( (x^*)^2 = 1 \), and \( x_0 > 0 \), the iteration formula implies that \( x_k > 0 \), so \( x^* = 1 \).

Since

\[ \lim_{x \to 1} \frac{|x_{k+1} - 1|}{|x_k - 1|^2} = \lim_{x \to 1} \frac{(2x_k + 1)(x_k - 1)^2}{3x_k^2(x_k - 1)^2} = \lim_{x \to 1} \frac{2x_k + 1}{3x_k^2} = 1. \]

By definition, we have the convergence order is 2.

4. Let \( H \in \mathbb{R}^{n\times n} \) be a symmetric matrix. Suppose

\[ a^T x + \frac{1}{2} x^T H x \geq 0 \]

for all \( x \in \mathbb{R}^n \).

Show that \( a = 0 \) and \( H \) is positive semidefinite.

\textit{Proof.} Let \( f(x) = a^T x + \frac{1}{2} x^T H x \), \( f(0) = 0 \), so for all \( x \in \mathbb{R}^n \), we have \( f(x) \geq f(0) \), which shows, \( x^* = 0 \) is global minimizer of \( f(x) \). By optimality condition, we have \( \nabla f(x^*) = a + H x^* = a = 0 \), and \( x^T H x \geq 0 \) for all \( x \in \mathbb{R}^n \), so by definition, we have \( H \) is positive semidefinite. \( \Box \)

5. Let \( f(x) : \mathbb{R}^n \to \mathbb{R} \) be a scalar valued twice differentiable function, and \( u \in \mathbb{R}^n \) be such that \( \nabla f(u) \neq 0 \). Suppose \( p^* \) is a minimizer of the quadratic function

\[ \varphi(p) = \nabla f(u)^T p + \frac{1}{2} p^T B p \]

where \( B \succ 0 \). Show that \( p^* \) is a descent direction \( f(x) \) at \( u \).

\textit{Proof.} \( p^* \) is a minimizer of the quadratic function, by optimality condition, we have

\[ \nabla f(u) + B p^* = 0, \]

and \( B \) is nonsingular, so

\[ p^* = -B^{-1} \nabla f(u). \]

\[ \nabla f(u)^T p^* = -\nabla f(u)^T B^{-1} \nabla f(u), B \succ 0, \text{ so } B^{-1} \succ 0, \text{ then } \nabla f(u)^T p^* < 0. \] so \( p^* \) is a descent direction at \( u \). \( \Box \)
6. Let \( A \in \mathbb{R}^{n \times n} \) be a nonsingular matrix, and \( f(x) : \mathbb{R}^m \to \mathbb{R} \) be a scalar valued function. Suppose \( \nabla f(0) = 0 \) and \( \nabla^2 f(0) \succ 0 \). Define the new function
\[
g(x) = f(Ax).
\]
Show that the origin is a strict local minimizer of \( g(x) \).

**Proof.** By chain rule, we have
\[
\nabla g(x) = A^T \nabla f(Ax), \quad \nabla^2 g(x) = A^T \nabla^2 f(Ax)A.
\]
At the origin, \( \nabla g(0) = 0 \), and \( \nabla^2 g(x) = A^T \nabla^2 f(0)A \). Since \( A \) is nonsingular and \( \nabla^2 f(0) \succ 0 \), we have \( \nabla^2 g(0) \succ 0 \). By optimality condition, 0 is a strict local minimizer of \( g(x) \). \( \square \)

7. Consider the quadratic optimization

\[
(QP) : \min x^T Ax \\
\text{s.t. } 1 - x^T x \geq 0,
\]
where \( A \in \mathbb{R}^{n \times n} \) is symmetric. Let \( x^* \) be a local minimizer and \( \lambda^* \) be its Lagrange multiplier. If \( \lambda_{\min}(A) + \lambda^* \geq 0 \), show that \( x^* \) is a global minimizer of this QP.

**Proof.** Let \( L(x) = x^T Ax - \lambda^*(1 - x^T x) \). Then
\[
\nabla L(x^*) = 0, \quad \nabla^2 L(x^*) = 2(A + \lambda^* I_n), \quad \lambda^* \geq 0, \quad \lambda^*(1 - (x^*)^T x^*) = 0.
\]
Note that \( L(x) \) is a quadratic function in \( x \) and its Hessian is positive semidefinite. So, \( x^* \) must be a global minimizer of \( L(x) \) in the entire space \( \mathbb{R}^n \), that is,
\[
L(x) - L(x^*) \geq 0 \quad \forall x \in \mathbb{R}^n.
\]
The above is same as
\[
x^T Ax - \lambda^*(1 - x^T x) - (x^*)^T Ax^* + \lambda^*(1 - (x^*)^T x^*) \geq 0 \quad \forall x \in \mathbb{R}^n.
\]
By optimality condition, \( \lambda^*(1 - (x^*)^T x^*) = 0 \). So, we get
\[
x^T Ax - (x^*)^T Ax^* \geq \lambda^*(1 - x^T x) \quad \forall x \in \mathbb{R}^n.
\]
Note that \( \lambda^* \geq 0 \) and \( 1 - x^T x \geq 0 \) when \( x \) is feasible for \( QP \). The above then implies that
\[
x^T Ax - (x^*)^T Ax^* \geq 0 \quad \forall x \text{ that is feasible for } (QP).
\]
By definition, \( x^* \) is a global minimizer of \( QP \). \( \square \)