Math 171B Homework Assignment #3 Solutions

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1. (10 points) Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a twice continuously differentiable function. Let \( \varphi(t) = f(u + td) \) be a composite function from \( \mathbb{R} \) to \( \mathbb{R} \), with given vectors \( u, d \in \mathbb{R}^n \). Express \( \varphi'(t) \) and \( \varphi''(t) \) in terms of the gradient and Hessian of \( f \) and the vectors \( u, d \).

Solution: Using the chain rule and the definitions of the gradient and Hessian, we get:

\[
\varphi'(t) = f'(u + td)d = d^T \nabla f(u + td)
\]

\[
\varphi''(t) = d^T \nabla^2 f(u + td)d
\]

2. (10 points) Decide whether or not the function \( e^{x_1^2 + x_2^2} \) is convex in the space \( \mathbb{R}^2 \).

Proof. Function \( f(x) = e^{x_1^2 + x_2^2} \) is continuous, then \( f(x) \) is convex iff \( \nabla^2 f(x) \succeq 0 \) for \( \forall x \in \mathbb{R}^2 \).

So calculate \( \nabla^2 f(x) = 2e^{x_1^2 + x_2^2} \begin{bmatrix} 1 + 2x_1^2 & 2x_1x_2 \\ 2x_1x_2 & 1 + 2x_2^2 \end{bmatrix} \), check it’s positive definite.

\[
1 + 2x_1^2 \geq 1, 1 + 2x_2^2 \geq 1, \begin{bmatrix} 1 + 2x_1^2 & 2x_1x_2 \\ 2x_1x_2 & 1 + 2x_2^2 \end{bmatrix} = 1 + 2x_1^2 + 2x_2^2 > 0. \]

So \( \nabla^2 f(x) \succeq 0 \), and \( f(x) \) is convex over \( \mathbb{R}^2 \).

Another Method
In order to check if \( H \) is positive definite here, we could use the following conclusion:

\[
\lambda_1 + \lambda_2 = \text{trace}(H)
\]

\[
\lambda_1 \lambda_2 = \det(H)
\]

But obviously, \( \text{trace}(H) \) and \( \det(H) \) are positive, so \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \), \( H \) is positive definite.

3. (10 points) Using the bisection method to construct a sequence of intervals \( \{ [a_k, b_k] \} \) approaching a zero point of \( \sin(\pi x) \) in the interval \( [-0.75, 2.25] \). Compute \( [a_k, b_k] \) for \( k = 1, 2, 3, 4 \). What is the limit of the sequence \( \{a_k\} \) and \( \{b_k\} \)?

Solution: \( \sin(\pi x) = 0 \) in the interval \( [-0.75, 2.25] \) have three zeros:

\[
x_1 = 0, x_2 = 1, x_3 = 2.
\]
Let \( f(x) = \sin(\pi x) \),

Use Bisection method:

step 1: \( a_1 = -0.75, \quad b_1 = 2.25, \quad f(a_1) = -0.7071, \quad f(b_1) = 0.7071, \quad c_1 = \frac{a_1 + b_1}{2} = 0.7500, \quad f(c_1) = 0.7071; \)

step 2: \( a_2 = -0.75, \quad b_2 = 0.7500, \quad f(a_2) = -0.7071, \quad f(b_2) = 0.7071, \quad c_2 = \frac{a_2 + b_2}{2} = 0, \quad f(c_2) = 0; \)

So we find one zero point, \( x^* = c_2 = 0 \). The algorithm converges in 2 steps.

4. (5 points) Use Newton’s method to solve the equation

\[ x - \frac{2}{x} = 0 \]

with the starting point \( x_0 = 1 \). Write down the Newton’s iteration formula, and compute \( x_1, x_2, x_3, x_4 \). If this Newton’s sequence converges, what is its limit?

Solution: \( f(x) = x - \frac{2}{x}, \quad f'(x) = 1 + \frac{2}{x^2}, \) Newton iteration:

\[
    x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}
\]

With \( x_0 = 1 \), we get:

\[
    x_1 = 1.3333, \quad x_2 = 1.4118, \quad x_3 = 1.4142, \quad x_4 = 1.4142,
\]

So if the algorithm converges, it is limit is \( x^* = \sqrt{2} \).

5. (5 points) Use Secant’s method to solve the equation

\[ x - \frac{2}{x} = 0 \]

with the starting point \( x_0 = 1, \quad x_1 = 2 \). Write down the iteration formula, and compute \( x_2, x_3, x_4, x_5 \).

Solution: \( x_{k+1} = x_k - \frac{f(x_k)(x_k-x_{k-1})}{f(x_k)-f(x_{k-1})} \)

\[
    x_2 = 1.5000, \quad x_3 = 1.4000, \quad x_4 = 1.4146, \quad x_5 = 1.4142.
\]

6. (10 points) Apply Newton’s method to solve equation \( x^3 - a = 0(a > 0) \), with initial guess \( x_0 > 0 \). If the Newton’s sequence converges, what is its convergence order? Justify your answer.

Solution: Newton method, convergence order \( =2 \), next we prove it. Newton method formula:

\[
    x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.
\]
\[ x^3 - a = 0 \text{ with } a > 0, \text{ solution } x^* = a^{\frac{1}{3}} > 0. \quad f'(x^*) = 3(x^*)^2 > 0, \quad f''(x^*) = 6x^* > 0, \text{ then} \]

\[
\lim_{k \to \infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|^2} = \frac{1}{2} \frac{f''(x^*)}{f'(x^*)} = \frac{1}{x^*} = \frac{1}{a^{1/3}} > 0.
\]

So it is convergence order is 2.

\textit{(Another Method)} In order to check the limit above, which is \(\frac{0}{0}\) type, L’Hospital rule could be used in this example. Consider:

\[
\lim_{k \to \infty} \frac{(x_{k+1} - x^*)}{(x_k - x^*)^2} = \lim_{k \to \infty} \frac{\frac{2x_k^3}{3x_k^3} - a^{1/3} + a}{\frac{2x_k^2}{3x_k} - 2a^{1/3}x_k + a^{2/3}} = \lim_{k \to \infty} \frac{2x_k^3 - 3a^{1/3}x_k^2 + a}{3x_k^2 - 6a^{1/3}x_k^2 + 3a^{2/3}x_k^2}
\]

\[
= \lim_{k \to \infty} \frac{6x_k^2 - 6a^{1/3}x_k}{12x_k^3 - 18a^{1/3}x_k^2 + 6a^{2/3}x_k} = \lim_{k \to \infty} \frac{x_k - a^{1/3}}{2x_k^2 - 3a^{1/3}x_k + a^{2/3}} = \lim_{k \to \infty} \frac{1}{4x_k - 3a^{1/3}} = \frac{1}{a^{1/3}}
\]

so convergence order is 2