1. (10 points) Show that for arbitrary $a, b$ the quadratic function

$$f(x_1, x_2) = x_1^2 + x_2^2 + x_1x_2 + ax_1 + bx_2$$

has a global minimizer. Show what values of $a, b$ the global minimum value of the above function is maximum?

Solution: There are two main approaches. The first is a straightforward, brute force calculation. The second avoids calculation until the very end.

Method 1.

Find the stationary point:

$$\nabla f = \begin{bmatrix} \nabla_{x_1} f \\ \nabla_{x_2} f \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 + a \\ 2x_2 + x_1 + b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

for any $a, b$, solve the above equations, we have

$$(x_1, x_2)^T = -\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} a \\ b \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 2a - b \\ 2b - a \end{bmatrix},$$

which is a stationary point. And calculate the hessian we have

$$\nabla^2 f = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

which is positive definite. So

$$(x_1, x_2)^T = -\frac{1}{3} \begin{bmatrix} 2a - b \\ 2b - a \end{bmatrix}$$

is a global minimizer for arbitrary $a, b$, bring it into $f(x_1, x_2)$ to get the global minimum value of $f$ is

$$f^*(x_1, x_2) = -\frac{1}{3} (a^2 + b^2 - ab).$$
which is a quadratic function with variable $a, b$, and to get the maximum value over $a, b$, we need to solve

$$\min_{a, b \in \mathbb{R}} g(a, b) = a^2 + b^2 - ab,$$

which is also a quadratic function.

Gradient

$$\nabla g(a, b) = \begin{bmatrix} 2a - b \\ 2b - a \end{bmatrix} = 0,$$

then $a^* = b^* = 0$, and

$$\nabla^2 g(a, b) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

which is positive semidefinite, so $(0, 0)$ is global minimizer of $g(a, b)$. The maximum of $f^*(x_1, x_2)$ is 0.

Method 2.

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $\lambda = \begin{bmatrix} a \\ b \end{bmatrix}$. Then we have:

$$f_\lambda(x) = \frac{1}{2}x^T Ax + \lambda^T x$$

$$\nabla f_\lambda(x) = Ax + \lambda$$

$$\nabla^2 f_\lambda(x) = A$$

Which gives us $x^* = -A^{-1}\lambda$ since $A$ is symmetric positive definite. Now define $g(\lambda) = f_\lambda(x^*) = -\frac{1}{2}\lambda^T A^{-1} \lambda$ and optimize $g$.

$$g(\lambda) = -\frac{1}{2}\lambda^T A^{-1} \lambda$$

$$\nabla g(\lambda) = -A^{-1} \lambda$$

$$\nabla^2 g(\lambda) = -A^{-1}$$

Which gives us $\lambda = 0$ is a maximizer since $-A^{-1}$ is symmetric negative definite (which implies it has trivial null space).

2. (10 points) Show that the following quadratic function

$$f(x_1, x_2) = x_1^2 + x_2^2 + 2x_1x_2 + x_1 + 2x_2$$

does not have a stationary point and is unbounded from below.

Solution: To get the stationary point, we need to solve the gradient equal to 0.

$$\nabla f(x_1, x_2) = \begin{bmatrix} 2x_1 + 2x_2 + 1 \\ 2x_2 + 2x_1 + 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
The coefficient matrix is singular, it is easy to see there is no solution, so function $f$ does not have a stationary point.

$$\nabla^2 f(x) = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix},$$

which is positive semidefinite and function $f(x)$ is convex. If function $f$ is bounded below, then there exists minimizer, which is a stationary point, and contradict there is no stationary point. So $f$ is unbounded below. See Lemma 3.2.1 in the book.

*Another method*

Here is another method to explain why this function is unbounded from below.

$$f(x_1, x_2) = x_1^2 + x_2^2 + 2x_1x_2 + x_1 + 2x_2 = (x_1 + x_2)^2 + x_1 + 2x_2$$

Consider the path $(x_1, x_2) = (t, -t)$, then $f(x_1, x_2) = -t$, when $t \to \infty$, $f \to -\infty$, which is unbounded from below.

3. (10 points) Let $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Show that if

$$x^T Ax + 2b^T x + c \geq 0, \ \forall x \in \mathbb{R}^n,$$

then the block matrix

$$\begin{bmatrix} A & b \\ b^T & c \end{bmatrix}$$

is positive semidefinite.

*Proof.* For any $x \in \mathbb{R}^n$, we have $x^T Ax + 2b^T x + c \geq 0,$

$$x^T Ax + 2b^T x + c = [x, 1] \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} [x, 1]^T \geq 0.$$

To prove $\begin{bmatrix} A & b \\ b^T & c \end{bmatrix}$ is positive semidefinite, we need to prove for any $y = [z, y_{n+1}] \in \mathbb{R}^{n+1}$ with $z \in \mathbb{R}^n$, we have

$$y^T \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} y = z^T Az + 2y_{n+1}b^T z + cy_{n+1}^2 \geq 0.$$

If $y_{n+1} \neq 0$, we have

$$y^T \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} y = y_{n+1}^2 \left( \frac{z}{y_{n+1}} A \frac{z}{y_{n+1}} + 2b^T \frac{z}{y_{n+1}} + c \right)$$
is arbitrary, by above condition, we have

\[ y^T \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} y \geq 0. \]

If \( y_{n+1} = 0 \), there exists sequence \( y^k \) with \( y^k_{n+1} \neq 0 \) converges to \( y \), and

\[ (y^k)^T \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} y^k \geq 0, \]

by continuity of this quadratic function, we must have

\[ y^T \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} y \geq 0. \]

\( y \) is arbitrary, so \( \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} \) is positive semidefinite.

Alternatively, one can note that the quadratic function \( x^T Ax + 2bx + c \) is bounded from below, and so its Hessian, \( A + A^T \) is positive semi-definite, which implies that \( A \) is positive semi-definite (why?). The case where \( y_{n+1} = 0 \) then follows immediately.

4. (10 points) Use golden section search to find a minimizer of function \( f(x) = x^4 + x \) in \([-1, 1]\) with 3 steps only. List all the obtained intervals containing the minimizer.

Solution: \( \tau = 0.618 \).

Step 1: \( a = -1, b = 1, x_1 = a + (1 - \tau) \cdot (b - a) = -0.2361, x_2 = a + \tau \cdot (b - a) = 0.2361, f(x_1) = -0.2330, f(x_2) = 0.2392. \)

Step 2: \( f(x_1) < f(x_2), \) so \( a = -1, b = 0.2361, x_2 = -0.2361, x_1 = a + (1 - \tau) \cdot (b - a) = -0.5279, f(x_1) = -0.4502, f(x_2) = -0.2330. \)

Step 3: \( f(x_1) < f(x_2), \) so \( a = -1, b = -0.2361, x_2 = -0.5279, x_1 = a + (1 - \tau) \cdot (b - a) = -0.7082, f(x_1) = -0.4566, f(x_2) = -0.4502. \)

5. (10 points) Let \( f(x) = \sin(\pi x/2) \). Find a cubic polynomial

\[ \varphi(x) = ax^3 + bx^2 + cx + d \]

such that \( \varphi(0) = f(0), \varphi'(0) = f'(0) \varphi(1) = f(1), \varphi'(1) = f'(1). \)

Solution:

(1) \( \varphi(0) = d = f(0) = \sin(0) = 0. \)
(2) $\phi'(0) = (3ax^2 + 2bx + c)|_{x=0} = c = f'(0) = \frac{\pi}{2}$.

(3) $\phi(1) = a + b + c + d = f(1) = 1$;

(4) $\phi'(1) = 3a + 2b + c = f'(1) = 0$;

This gives us the linear system:

$$
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 \\
3 & 2 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c \\
d
\end{bmatrix} =
\begin{bmatrix}
0 \\
\pi/2 \\
1 \\
0
\end{bmatrix}
$$

Which has the unique solution:

$$
\begin{bmatrix}
a \\
b \\
c \\
d
\end{bmatrix} =
\begin{bmatrix}
\pi/2 - 2 \\
3 - \pi \\
\pi/2 \\
0
\end{bmatrix}
$$