Math 171B Homework Assignment #7 Solutions

1. (6 points) Find all the local minimizers of

\[
\min x_1x_2 + x_2x_3 + x_3x_1 \\
\text{s.t. } x_1^2 + x_2^2 + x_3^2 = 1.
\]

Which one of them is a global minimizer?

Solution: Constraint qualification condition.

\[
\nabla f(x) = \lambda \nabla c(x).
\]

So

\[
\begin{bmatrix}
  x_2 + x_3 \\
  x_1 + x_3 \\
  x_1 + x_2
\end{bmatrix}
= \lambda 
\begin{bmatrix}
  2x_1 \\
  2x_2 \\
  2x_3
\end{bmatrix}.
\]

Solve the equation, we have \( \lambda = 1 \) or \( x_1 + x_2 + x_3 = 0 \).

If \( \lambda = 1 \), we have \( x_1 = x_2 = x_3 \), bring back to \( c(x) \) and solve the equation:

\[
x_1 = x_2 = x_3 = \pm \sqrt{\frac{1}{3}}.
\]

\[
J_c(x^*) = \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 \end{bmatrix}, \text{ for } p \in \text{Null}(J_c(x^*)), \text{ check the second order condition}
\]

\[
\nabla^2 f(x) - \lambda \nabla^2 c(x) = \begin{bmatrix} 0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix} - \lambda \begin{bmatrix} 2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{bmatrix}.
\]

since \( \lambda = 1 \), we can not have second order condition is true.

So \( x_1 + x_2 + x_3 = 0 \), with condition \( x_1^2 + x_2^2 + x_3^2 = 1 \), we have: \( \lambda = -\frac{1}{2} \),

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \alpha \begin{bmatrix}
\frac{1}{\sqrt{3}} \\
-\frac{1}{\sqrt{2}} \\
0
\end{bmatrix} + \beta \begin{bmatrix}
\frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} \\
-\frac{2}{\sqrt{6}}
\end{bmatrix}, \text{ s.t. } \alpha^2 + \beta^2 = 1.
\]

Notice that at each of these points, we have:
\[ x_1 x_2 + x_2 x_3 + x_3 x_1 = \left( \frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{6}} \right) \left( -\frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{6}} \right) + \left( -\frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{6}} \right) \left( \frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{6}} \right) \left( -\frac{2\beta}{\sqrt{6}} \right) + \left( \frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{6}} \right) \left( -\frac{2\beta}{\sqrt{6}} \right) \]
\[ = \left( \frac{\beta^2}{6} - \frac{\alpha^2}{2} \right) - \frac{4\beta^2}{6} \]
\[ = -\frac{1}{2} (\alpha^2 + \beta^2) \]
\[ = -\frac{1}{2} \]

Check second order condition:

\[ H = \nabla^2 f(x) - \lambda \nabla^2 c(x) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \succeq 0. \]

so for \( p \in \text{Null}(J_c(x^*)) \), the second order necessary condition is satisfied. Further, since in the direction \( p = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \), \( p^T Hp = 9 > 0 \), the second order condition is strictly satisfied, and the object function is constant along the perpendicular direction as shown above, we can conclude that all these points are local and global minimizers (but they are neither strict nor isolated!).

2. (6 points) Let \((a, b) \neq (0, 0)\) be a real pair. Find a local maximizer of

\[ \max ax_1 + bx_2 \]
\[ \text{s.t. } x_1^4 + x_2^4 = 1. \]

Is it also a global maximizer?

Solution: Consider the following equivalent minimization problem:

\[ \min -ax_1 - bx_2 \]
\[ \text{s.t. } x_1^4 + x_2^4 = 1. \]

First order condition:

\[ \begin{bmatrix} -a \\ -b \end{bmatrix} = 4\lambda \begin{bmatrix} x_1^3 \\ x_2^3 \end{bmatrix} \]

bring into \( x_1^4 + x_2^4 = 1 \), we get two solutions:

\[ \lambda_1 = \frac{1}{4} (a^4 + b^4)^{\frac{3}{2}}, \quad x_1 = \left( \frac{-a}{4\lambda_1} \right)^{\frac{1}{2}}, x_2 = \left( \frac{-b}{4\lambda_1} \right)^{\frac{1}{2}} \]

Or
\[
\lambda_2 = -\frac{1}{4}(a^\frac{1}{4} + b^\frac{1}{4})^4, \quad x_1 = \left(-\frac{a}{4\lambda_2}\right)^\frac{1}{4}, x_2 = \left(-\frac{b}{4\lambda_2}\right)^\frac{1}{4}
\]

check second order condition
\[
\nabla^2 f(x) - \lambda \nabla^2 c(x) = -\lambda \begin{bmatrix} 12x_1^2 & 0 \\ 0 & 12x_2^2 \end{bmatrix},
\]
which is positive semidefinite if \(\lambda \leq 0\), so the second solution is global minimizer, which is the global maximizer of the original problem.

3. (8 points) Find all the local minimizer of
\[
\min x_1^2 + x_2^2 + 4x_1x_2 \\
\text{s.t. } x_1^2 \leq 1, x_2^2 \leq 1.
\]
Identify the global minimizers of them.

Solution. By optimality condition, we need to decide which constraint is active.

There are three cases:

(1) first inequality constraint is active and second one is not active, then by first order optimality condition
\[
\nabla f(x) = \lambda \nabla c_1(x).
\]
If \(c_1(x)\) is active, then \(x_1^\star = \pm 1\), so
\[
\begin{bmatrix} 2x_1 + 4x_2 \\ 4x_1 + 2x_2 \end{bmatrix} = \lambda \begin{bmatrix} -2x_1 \\ 0 \end{bmatrix}.
\]
then \(x_1^\star = 1, x_2^\star = -2, \lambda^\star = 3\), and \(x_1^\star = -1, x_2^\star = 2, \lambda^\star = 3\), but \(x_2^\star\) is not feasible.

(2) second inequality constraint is active and the first one is not active, then by first order optimality condition
\[
\nabla f(x) = \lambda \nabla c_2(x).
\]
If \(c_2(x)\) is active, then \(x_2^\star = \pm 1\), so
\[
\begin{bmatrix} 2x_1 + 4x_2 \\ 4x_1 + 2x_2 \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ -2x_2 \end{bmatrix}.
\]
then \(x_1^\star = -2, x_2^\star = 1, \lambda^\star = 3\), and \(x_1^\star = 2, x_2^\star = -1, \lambda^\star = 3\), the first inequality is not feasible.

(3) two inequalities are active, then there are four feasible points \(x_1 = \pm 1, x_2 = \pm 1\).
At (1,1), check
\[
\begin{bmatrix}
2x_1 + 4x_2 \\
4x_1 + 2x_2
\end{bmatrix} = \lambda_1 \begin{bmatrix}
-2x_1 \\
0
\end{bmatrix} + \lambda_2 \begin{bmatrix}
0 \\
-2x_2
\end{bmatrix}
\]
\[
\begin{bmatrix}
6 \\
6
\end{bmatrix} = \lambda_1 \begin{bmatrix}
-2 \\
0
\end{bmatrix} + \lambda_2 \begin{bmatrix}
0 \\
-2
\end{bmatrix}, \quad \lambda_1 = \lambda_2 = -3.
\]
so first order condition is not satisfied, and (1,1) is not minimizer.

At (-1, -1),
\[
\begin{bmatrix}
-6 \\
-6
\end{bmatrix} = \lambda_1 \begin{bmatrix}
2 \\
0
\end{bmatrix} + \lambda_2 \begin{bmatrix}
0 \\
2
\end{bmatrix}, \quad \lambda_1 = -3, \lambda_2 = -3.
\]
first order condition is not satisfied.

At (1,-1),
\[
\begin{bmatrix}
-2 \\
-2
\end{bmatrix} = \lambda_1 \begin{bmatrix}
-2 \\
0
\end{bmatrix} + \lambda_2 \begin{bmatrix}
0 \\
2
\end{bmatrix}, \quad \lambda_1 = 1, \lambda_2 = 1.
\]

At (-1, 1),
\[
\begin{bmatrix}
2 \\
-2
\end{bmatrix} = \lambda_1 \begin{bmatrix}
2 \\
0
\end{bmatrix} + \lambda_2 \begin{bmatrix}
0 \\
-2
\end{bmatrix}, \quad \lambda_1 = 1, \lambda_2 = 1.
\]

Check the second order condition for (1,-1), (-1,1),
\[
\nabla^2 f(x) - \lambda_1 \nabla^2 c_1(x) - \lambda_2 \nabla^2 c_2(x) = \begin{bmatrix}
2 & 4 \\
4 & 2
\end{bmatrix} - \begin{bmatrix}
-2 & 0 \\
0 & 0
\end{bmatrix} - \begin{bmatrix}
0 & 0 \\
0 & -2
\end{bmatrix} = \begin{bmatrix}
4 & 4 \\
4 & 4
\end{bmatrix} \succeq 0.
\]
so for all \( p \in \text{Null}(J_a(x^*)) \), we have \( p^T H(x^*, \lambda^*) p \geq 0 \), second order condition is true, and we find two minimizers (1,-1), (-1,1), with minimum \( f(-1,1) = f(1,-1) = -2 \).

4. (10 points) Find the point on the parabola \( 5y = (x - 1)^2 \) that is closest to point (1,2). Formulate this problem as a constrained optimization, and then solve it by using optimality conditions.

Solution. The closest to point to (1, 2), we formulate it as an optimization problem:
\[
\begin{align*}
\min \quad & (x - 1)^2 + (y - 2)^2 \\
\text{s.t.} \quad & (x - 1)^2 - 5y = 0.
\end{align*}
\]
first order necessary condition \( \nabla f(x) = \lambda \nabla c(x) \), so
\[
\begin{bmatrix}
2(x - 1) \\
2(y - 2)
\end{bmatrix} = \lambda \begin{bmatrix}
2(x - 1) \\
5
\end{bmatrix},
\]
With constraint \( (x - 1)^2 = 5y \), we have \( x^* = 1, y^* = 0, \lambda = \frac{4}{5} \).
Second order condition
\[ \nabla^2 f(x) - \lambda \nabla^2 c(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \lambda^* \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \succeq 0. \]

So for any \( p \in \text{Null}(\nabla c(x)) \), second order condition is satisfied. so \((1, 0)\) is the point on the parabola that is closest to point \((1, 2)\).

5. (10 points) Let \( A \in \mathbb{R}^{n \times n} \) be symmetric. If \( \lambda \) is a Lagrange multiplier of
\[
\begin{align*}
\min & \quad x^T Ax \\
\text{s.t.} & \quad x^T x - 1 = 0,
\end{align*}
\]
show that \( \lambda \) is an eigenvalue of \( A \).

Proof. \( A \) is symmetric, and \( \lambda \) is a Lagrange multiplier, by optimality condition, we have \( \nabla f(x) = \lambda \nabla c(x) \), so
\[ Ax = \lambda x. \]

So \( \lambda \) is an eigenvalue of \( A \). \( \square \)

6. (10 points) Let \( x^* \) be a local minimizer of
\[
\begin{align*}
\min & \quad g^T x + x^T H x \\
\text{s.t.} & \quad 1 - x^T x \geq 0,
\end{align*}
\]
where \( g \in \mathbb{R}^n \) and \( H \in \mathbb{R}^{n \times n} \) is symmetric. Suppose \( \lambda^* \) is its Lagrange multiplier. If \( H + \lambda^* I_n \succeq 0 \), show that \( x^* \) is also a global minimizer.

Solution: \( \lambda^* \) is its Lagrange multiplier, \( \lambda^* \succeq 0 \).

first order condition
\[
2Hx^* + g = -2\lambda^* x^*, \text{so } g = -2\lambda^* x^* - 2H x^*, \text{ and } \lambda^*(1 - (x^*)^T x^*) = 0
\]
\[
f(x) = x^T H x + g^T x = (x^* - (x^* - x))^T H (x^* - (x^* - x)) + g^T (x^* - (x^* - x)).
\]
Let \( p = x^* - x \),
\[
f(x) = (x^* - p) H (x^* - p) + g^T (x^* - p) = f(x^*) - 2p^T H x^* + p^T H p - g^T p.
\]
So \( f(x) - f(x^*) = -2p^T H x^* + p^T H p - g^T p = p^T (H + \lambda^* I_n)p - 2p^T H x^* - \lambda^* p^T p - g^T p \)
By above \( g \), we have \( g^T p = -2\lambda^* (x^*)^T p - 2p^T H x^* \),
\[
f(x) - f(x^*) = p^T (H + \lambda^* I_n)p - 2p^T H x^* - \lambda^* p^T p + 2\lambda^* (x^*)^T p + 2p^T H x^* \]
\[= p^T (H + \lambda^* I_n)p - \lambda^* p^T p + 2\lambda^* (x^*)^T p \]
\[-\lambda^* p^T p + 2\lambda^* (x^*)^T p = -\lambda^* [p^T p - 2(x^*)^T p + (x^*)^T x^*] + \lambda^* (x^*)^T x^* \]
\[= -\lambda^* (p - x^*)^T (p - x^*) + \lambda^* (x^*)^T x^* \]
\[= -\lambda^* x^T x + \lambda^* (x^*)^T x^* \]

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As $\lambda^*(1 - (x^*)^T x^*) = 0$, so $\lambda^* = \lambda^*(x^*)^T x^*$, bring into above equation, we have

$$f(x) - f(x^*) = p^T (H + \lambda^* I_n)p - \lambda^* p^T p + 2\lambda^* (x^*)^T p$$
$$= p^T (H + \lambda^* I_n)p - \lambda^* x^T x + \lambda^*$$
$$= p^T (H + \lambda^* I_n)p + \lambda^* (1 - x^T x)$$

$H + \lambda^* I \succeq 0$, and $x$ is feasible, so $1 - x^T x \geq 0$, $\lambda^* \geq 0$.
So for any $x \in X$, $f(x) - f(x^*) \geq 0$, which proves $x^*$ is global minimizer.