Solutions for Practice Midterm Math 100b

1. If \( x = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \) and \( y = \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \) then \( x+y = \begin{bmatrix} a+c & b+d \\ -(b+d) & a+c \end{bmatrix} \) \( \in \) \( R \) and
\[
x y = \begin{bmatrix} ac - bd & ad + bc \\ -bc - ad & -bd + ac \end{bmatrix} = y x \in R.
\]

So \( R \) is a subring. If \( z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \) then \( x z = \begin{bmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{bmatrix} = x z \).

Suppose that \( F \) is totally real if \( x \neq 0 \) then \( a^2 + b^2 \neq 0 \) and we can set
\[
w = \begin{bmatrix} \frac{a}{a^2 + b^2} & \frac{-b}{a^2 + b^2} \\ \frac{b}{a^2 + b^2} & \frac{-a}{a^2 + b^2} \end{bmatrix}
\]
and find that \( w x = x w = I \). So \( R \) is a field. If \( R \) is not totally real then there exist \( a, b \) not both \( 0 \) such that \( a^2 + b^2 = 0 \). With that choice of \( a, b \) and \( x, z \) as above we have \( x z = 0 \). Since \( x \neq 0 \) and \( z \neq 0 \) we see that \( R \) is not an integral domain so not a field. Thus \( R \) is a field if and only if \( F \) is totally real.

2. (a) This is not a group homomorphism since \( \phi(1 + 1) = \phi(2) = 2^2 = 4 \) but \( \phi(1) + \phi(1) = 1 + 1 = 2 \).

(b) Let \( u(x) = a_0 + a_1 x + \ldots + a_n x^n \) and \( v(x) = b_0 + b_1 x + \ldots + b_m x^m \). Let \( p \) be the maximum of \( m \) and \( n \). We define \( a_j = 0 \) if \( j > n \) and \( b_j = 0 \) if \( j > n \). Then
\[
u(x) \pm v(x) = (a_0 \pm b_0) + (a_1 \pm b_1)x + \ldots + (a_p \pm b_p)x^p.
\]
Thus
\[
\phi(u(x) \pm v(x)) = f(a_0 \pm b_0) + \ldots + f(a_p \pm b_p)x^p
\]
\[
= (f(a_0) \pm f(b_0)) + \ldots + (f(a_p) \pm f(b_p))x^p
\]
\[
= (f(a_0) + f(a_1)x + \ldots + f(a_p)x^p) \pm (f(b_0) + f(b_1)x + \ldots + f(b_p)x^p)
\]
\[
= \phi(u(x)) \pm \phi(v(x)).
\]
Thus \( \phi \) is a group homomorphism under addition. Also
\[
u(x)v(x) = \sum_{k=0}^{2p} \left( \sum_{i=0}^{k} a_ib_{k-i} \right) x^k.
\]
So
\[
\phi(u(x)v(x)) = \sum_{k=0}^{2p} f \left( \sum_{i=0}^{k} a_ib_{k-i} \right) x^k
\]
\[
= \sum_{k=0}^{2p} \left( \sum_{i=0}^{k} f(a_i)b_{k-i} \right) x^k = \sum_{k=0}^{2p} \left( \sum_{i=0}^{k} f(a_i)f(b_{k-i}) \right) x^k
\]
\[
\phi(\phi(u(x))\phi(v(x))).
\]

Thus \( \phi \) is a ring homomorphism.

(c) If \([a], [b] \in \mathbb{Z}_3\) then

\[
\phi([a] \pm [b]) = \phi([a \pm b])
\]

\[
= [(a \pm b)^3] = [a^3 \pm 3a^2b + 3ab^2 \pm b^3]
\]

\[
= [a^3 \pm b^3] = [a^3] \pm [b^3] = \phi([a]) \pm \phi([b]).
\]

Also \( \phi([a][b]) = \phi([ab]) = [(ab)^3] = [a^3b^3] = [a^3][b^3] = \phi(a)\phi(b) \). Thus \( \phi \) is a ring homomorphism.

3. We note that the subring \( R = \{2n|n \in \mathbb{Z}\} \) of \( \mathbb{Z} \) is an ideal in itself. If \( R \subset R \) were a principal ideal in \( R \) then there would exist \( a \in R \) such that \( R = \{ar|r \in R\} \). In particular, there would be an \( r \in R \) such that \( a = ar \). Since \( R \neq \{0\} \), \( a \neq 0 \) so cancelling the \( a \) in \( \mathbb{Z} \) yields \( r = 1 \). But \( 1 \notin R \) since the ring \( R \) is a proper ideal of \( \mathbb{Z} \). We have thus shown that the ideal \( R \) in \( R \) is not principal.

4. We have

\[
1 + x^2 + x^4 - \frac{1}{3}x(1 + 4x + 4x^2 + 3x^3) = 1 - \frac{1}{3}x - \frac{1}{3}x^2 - \frac{4}{3}x^3.
\]

\[
1 + 4x + 4x^2 + 3x^3 + \frac{9}{4}(1 - \frac{1}{3}x - \frac{1}{3}x^2 - \frac{4}{3}x^3) = \frac{13}{4} + \frac{13}{4}x + \frac{13}{4}x^2.
\]

\[
1 + 4x + 4x^2 + 3x^3 - 3x(1 + x + x^2) = 1 + x + x^2.
\]

A greatest common divisor is therefore \( 1 + x + x^2 \).