Problems

1. If $U$ is an ideal of $R$ and $1 \in U$, prove that $U = R$.

2. If $F$ is a field, prove its only ideals are $(0)$ and $F$ itself.

3. Prove that any homomorphism of a field is either an isomorphism or takes each element into 0.

4. If $R$ is a commutative ring and $a \in R$,
   (a) Show that $aR = \{ar \mid r \in R\}$ is a two-sided ideal of $R$.
   (b) Show by an example that this may be false if $R$ is not commutative.

5. If $U, V$ are ideals of $R$, let $U + V = \{u + v \mid u \in U, v \in V\}$. Prove that $U + V$ is also an ideal.

6. If $U, V$ are ideals of $R$ let $UV$ be the set of all elements that can be written as finite sums of elements of the form $uv$ where $u \in U$ and $v \in V$. Prove that $UV$ is an ideal of $R$.

7. In Problem 6 prove that $UV \subseteq U \cap V$.

8. If $R$ is the ring of integers, let $U$ be the ideal consisting of all multiples of 17. Prove that if $V$ is an ideal of $R$ and $R \supseteq V \supseteq U$ then either $V = R$ or $V = U$. Generalize!
9. If $U$ is an ideal of $R$, let $r(U) = \{x \in R \mid xu = 0 \text{ for all } u \in U\}$. Prove that $r(U)$ is an ideal of $R$.

10. If $U$ is an ideal of $R$ let $[R:U] = \{x \in R \mid rx \in U \text{ for every } r \in R\}$. Prove that $[R:U]$ is an ideal of $R$ and that it contains $U$.

11. Let $R$ be a ring with unit element. Using its elements we define a ring $\tilde{R}$ by defining $a \oplus b = a + b + 1$, and $a \cdot b = ab + a + b$, where $a, b \in R$ and where the addition and multiplication on the right-hand side of these relations are those of $R$.
   (a) Prove that $\tilde{R}$ is a ring under the operations $\oplus$ and $\cdot$.
   (b) What acts as the zero-element of $\tilde{R}$?
   (c) What acts as the unit-element of $\tilde{R}$?
   (d) Prove that $R$ is isomorphic to $\tilde{R}$.

*12. In Example 3.1.6 we discussed the ring of rational $2 \times 2$ matrices. Prove that this ring has no ideals other than $(0)$ and the ring itself.