1. Let $G = \left\{ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z}_3 \text{ and } ad - bc = 1 \right\}$.

(a) Prove that $G$ has order 24. Hint: Set $H = \left\{ g = \begin{bmatrix} T & x \\ 0 & T \end{bmatrix} \mid x \in \mathbb{Z}_3 \right\}$.
Show that $\phi : G/H \to \{(x, y) \mid (x, y) \neq (0, 0)\}$ given by $\phi(gH) = (a, c)$ for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is well defined and bijective. (Hint: Show that if $m = \begin{bmatrix} a & b' \\ c & d' \end{bmatrix}$ and $ad' - b'c = 1$ then $g^{-1}m \in H$ so $m \in gH$.)

(b) Prove that $G$ is not isomorphic with $S_4$.

2. Let $n \geq 4$ consider the group $H$ generated by all 4-cycles in $S_n$. Show that $H = S_n$. Hint: First show that $H$ is a normal subgroup. Then show that $(1432)(1423)(1243) = (12)$.

3. Which of the following sets with binary operations as given (the $a$ operation is for the addition the $m$ for multiplication) is a ring with the indicated 0 and 1? You must give reasons to get full credit.
   a) $R = \mathbb{Z}$ and $a(x, y) = x - y$ and $m(x, y) = xy$ with $0_R = 0$ and $1_R = 1$.
   b) $R = \mathbb{Z}$ and $a(x, y) = x + y + 1$ and $m(x, y) = xy + x + y$ with $0_R = -1$ and $1_R = 0$.
   c) $R = \mathbb{R} \times \mathbb{R}$ and $a((x_1, x_2), (y_1, y_2)) = (x_1 + y_1, x_2 + y_2)$, $m((x_1, x_2), (y_1, y_2)) = (x_1y_1 + x_2y_2, x_1y_2 + x_2y_1)$ with $0_R = (0, 0), 1_R = (1, 0)$.

4. Let $G$ be a commutative group and let $H$ denote the set of elements of finite order in $G$.
   a) Show that $H$ is a subgroup of $G$.
   b) Consider the matrices $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ calculate the orders of $A$ and $B$ and deduce that the elements are of finite order. Show that $AB$ does not have finite order. Why doesn’t this contradict part a)?

5. Assume that $G$ is a group whose order is 10 show that $G$ is isomorphic to ether $\mathbb{Z}_{10}$ or $D_5$ (the dihedral group with 10 elements).

6. Let $G$ be a group and let $H$ be a cyclic subgroup of $G$ that is normal in $G$ show that every subgroup of $H$ is normal in $G$.

7. Let $R$ be a commutative ring and $S$ be the set of $2 \times 2$ matrices with entries in $R$ under matrix multiplication. Show that $S$ is a ring. Show that $S^* = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in R \text{ and } ad - bc \in R^* \right\}$.
8. Let $G$ be a finite group such that if $g \in G$ then $g^2 = e$ (the identity element).
   a) Prove that $G$ is abelian.
   b) Prove that $|G| = 2^n$ for some $n$.
   c) Prove that if $|G| = 2^n$ then $G$ is isomorphic with the product group $C_2 \times C_2 \times \cdots \times C_2$ ($n$-copies).
   Hint: Prove by induction on $n$ that if $|G| = 2^n$ then there exists an onto group homomorphism from $G$ to $C_2$. (To do this observe that if $x \in G$ and $x \neq e$ then $H = \{e, x\}$ is a normal subgroup and $G/H$ has order $2^{n-1}$. If there exists an onto homomorphism, $\phi$, of $G/H$ onto $C_2$ then $\eta(g) = \phi(gH)$ is an onto homomorphism of $G$ to $C_2$.) Write $C_2 = \{e, a\}$. Let $\eta$ be an onto homomorphisms of $G$ to $C_2$ and let $U$ be the kernel of $\eta$. If $x \in G$ is such that $\eta(x) = a$ then show that the map $\alpha : C_2 \times U \to G$ given by $\alpha(e, u) = u$ and $\alpha(a, u) = xu$ defines a group homomorphism.