1. a) This relation is symmetric and reflexive but not transitive. To show that it is not transitive notice that \( a = 0 \) for all \( a \). Thus \( 2 = 0 \) and \( 3 = 0 \) so \( 0 \cdot 3 = 0 \) hence \( 0 \cdot 3 = 0 \). But \( 2 \neq 3 \) and \( 2 \cdot 3 \neq 0 \).

b) Reflexive: \( a = a \) hence \( a = a \).

Symmetric: \( a = b \) implies that \( a = b \) or \( ab = 1 \). This implies that \( b = a \) or \( ba = 1 \).

Transitive: \( a = b \) and \( b = c \) implies that \( a = b \) or \( ab = 1 \) and \( b = c \) or \( bc = 1 \). If \( a = b \) and \( b = c \) then \( a = c \) so \( a = c \).

Thus the relation is an equivalence relation.

c) Reflexive: \( s^2 = t^2 \).

Symmetric: If \( s^2 = t^2 \) then \( t^2 = s^2 \).

Transitive: If \( s^2 = t^2 \) and \( t^2 = u^2 \) then \( s^2 = u^2 \).

The equivalence class of \( t \) consists of all \( s \) with \( s^2 = t^2 \). If \( t \neq 0 \) then the equation \( x^2 = t^2 \) has two solutions \( x = t \) and \( x = -t \) if \( t = 0 \) then the only solution is \( x = 0 \). Thus the set of equivalence classes is \( \{ \{0\} \} \cup \{ \{t,-t\} | t > 0 \} \).

2. a) Suppose that \( a \in \mathbb{Z} \) and \( a + x + 2 = x \) for all \( x \in \mathbb{Z} \). Then solving for \( a \) we have \( a = -2 \). Thus there is an identity, \( e = -2 \). If \( a, b, c \in \mathbb{Z} \) then

\[
a \ast (b \ast c) = a + b \ast c + 2 = a + (b + c + 2) + 2 = a + b + c + 4.
\]

Also

\[
(a \ast b) \ast c = (a \ast b) + c + 2 = (a + b + 2) + c + 2 = a + b + c + 4.
\]

Thus this binary operation is associative. Suppose \( a \in \mathbb{Z} \) then we attempt to solve \( a \ast x = -2 \). That is \( a + x + 2 = -2 \). Then \( x = -a - 4 \). Since \( a + (-a - 4) + 2 = -2 \) we can take the inverse of \( a \) to be \(-a - 4 \). We have proved that this example is a group.

b) Suppose that \( a = x \ast a \) for all \( a \in \mathbb{Z} \). Then \( xa + 2x + 2a + 2 = a \) for all \( a \). Hence

\[
x(a + 2) = -(a + 2).
\]

One solution is \( x = -1 \). Thus we can take \( e = -1 \). As for the associative rule we have

\[
a \ast (b \ast c) = a(b \ast c) + 2a + 2b \ast c + 2 =
\]

\[
a(bc + 2b + 2c + 2) + 2a + 2(bc + 2b + 2c + 2) + 2 =
\]

\[
abc + 2ab + 2ac + 2a + 2bc + 4b + 4c + 4 + 2 =
\]

\[
abc + 2ab + 2ac + 2bc + 4a + 4b + 4c + 6.
\]

We now calculate

\[
(a \ast b) \ast c = (ab + 2a + 2b + 2) \ast c =
\]

\[
(ab + 2a + 2b + 2)c + 2(ab + 2a + 2b + 2)c + 2 =
\]

\[
abc + 2ac + 2bc + 2 + 2ab + 4a + 4b + 4 + 2c + 2 =
\]

\[
abc + 2ac + 2bc + 2ab + 4a + 4b + 4c + 6.
\]

The two expressions are equal so the associative rule is true. Now suppose that \( a \in \mathbb{Z} \) then an inverse of \( a \) must satisfy \( x \ast a = -1 \). That is \( xa + 2a + 2x = -1 \). That is

\[
x(a + 2) = -2a - 1.
\]

If \( a = -2 \) then this says \( x \ast 0 = 3 \). We can’t solve this equation so in this case we only have a monoid.
c) An identity element must satisfy \( x \cdot a = a \) for all \( a \). That is \( x^2 \cdot a = a \) for all \( a \). If \( a \neq 0 \) then this says that \( x^2 \cdot 1 = 1 \) for all \( a \). There is no such \( x \) since if \( a = -1 \) then there would have to be an \( x \in \mathbb{Z} \) with \( x^2 = -1 \). Relative to this binary operation \( Z \) is not a monoid.

3. In \( \mathbb{Z}_{17} \), \( \bar{1}6 = (-1) \). We observe that if \( a \in \mathbb{Z} \) then in \( \mathbb{Z}_n \), \( a^n = \bar{a}^n \). We prove this by induction on \( n \). If \( n = 1 \) then \( \bar{a}^{-1} = \bar{a} = a^{-1} \). Assume for \( n = m \). Then \( \bar{a}^{m+1} = \bar{a}^m \bar{a}^{-1} = \bar{a}^m \bar{a} = \bar{a}^{m+1} \). The assertion is thus true for all \( n \). In the case at hand this says that \( \bar{1}6^{27} = (-1)^{27} = (-1) \).

4. \( Z_8 = \langle \bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7} \rangle \). It’s Cayley table under multiplication is

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
2 & 0 & 2 & 4 & 6 & 0 & 2 & 4 \\
3 & 0 & 3 & 6 & 1 & 4 & 7 & 2 \\
4 & 0 & 4 & 0 & 4 & 0 & 4 & 0 \\
5 & 0 & 5 & 2 & 7 & 4 & 1 & 6 \\
6 & 0 & 6 & 4 & 2 & 0 & 6 & 4 \\
7 & 0 & 7 & 6 & 5 & 4 & 3 & 2 \\
\end{array}
\]

We note that the rows (or columns) that contain \( \bar{1} \) are labeled by \( \bar{1}, \bar{3}, \bar{5}, \bar{7} \). These are therefore the invertible elements. By deleting the row and column for the other elements we get the Cayley table for \( Z_8^* \).

\[
\begin{array}{cccc}
1 & 3 & 5 & 7 \\
\hline
\bar{1} & \bar{1} & \bar{3} & \bar{5} \\
\bar{3} & \bar{3} & \bar{1} & \bar{7} \\
\bar{5} & \bar{5} & \bar{7} & \bar{1} \\
\bar{7} & \bar{7} & \bar{5} & \bar{3} \\
\end{array}
\]

Consider \( a = \bar{3}, b = \bar{5} \) then \( a^0 b^0 = \bar{1} \cdot \bar{1} = \bar{1}, a^1 b^0 = \bar{a} \cdot \bar{1} = \bar{a} = \bar{3}, a^0 b^1 = \bar{b} = \bar{5}, a^1 b^1 = \bar{a} \cdot \bar{5} = \bar{5} \cdot \bar{5} = \bar{7} \). This gives every element. Any of the three pairs \( \{\bar{3}, \bar{5}\}, \{\bar{3}, \bar{7}\}, \{\bar{5}, \bar{7}\} \) would have worked.

5. If \( ab = ba \) and \( a^2 = b^2 = e \) (the identity element) then applying the associative rule we have

\[
(ab)^2 = (ab)(ab) = ((ab)a)b.
\]

Applying it again we have

\[
((ab)a)b = (a(ba))b.
\]

we now use the fact that \( ba = ab \) so

\[
(a(ba))b = (a(ab))b.
\]

Apply the associative rule again and have \( ((ab)b)b = bb = e \).

The following argument would have gotten full credit for this part since we already know that it doesn’t matter how we place the parentheses in the multiplication. \( (ab)^2 = abab = a(ba)b = a(ab)b = aabb = e. \)

\[
g = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}, h = \begin{bmatrix}
1 & 0 \\
1 & -1
\end{bmatrix}. \text{ Then } g^2 = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \text{ and}
\]
\[ h^2 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \text{ Now} \]

\[ gh = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \]

so \((gh)^2 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \). \]

\((gh)^3 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \]

\((gh)^4 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}. \]

\((gh)^5 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}. \]

\((gh)^6 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \]

This is not a contradiction to the first part of the problem since \(gh \neq hg\).