1. We use Fibonacci’s method \( \frac{7}{2} \geq \frac{1}{2} \) and \( \frac{1}{2} \geq \frac{1}{2} \). So \( \frac{7}{2} = \frac{1}{2} + \frac{5}{2} \). For \( \frac{5}{11} \) the method says start with \( \frac{1}{11} \). Then \( \frac{5}{11} - \frac{1}{11} = \frac{1}{2} \). Next we take \( 1 \) and \( \frac{2}{1} - \frac{1}{1} = \frac{2}{1} \). So \( \frac{2}{11} = \frac{1}{1} + \frac{1}{11} \). As for the sexagesimal we start by getting one digit that is try \( \frac{2}{11} = \frac{2}{11} \). Then \( 7a = 120 \).

Since \( 7 \times 17 = 119 \) we see that \( \frac{2}{11} = \frac{17}{110} + \frac{1}{110} \). We thus want \( \frac{17}{110} = \frac{b}{110} \). That is \( 7b = 60 \) thus the next digit is \( 8 \). So \( \frac{5}{11} = ; 17, 8, \ldots \).

For \( \frac{5}{11} \) we start as before \( \frac{15}{11} = \frac{a}{110} \) so \( 11a = 300 \). The first digit is thus \( 27 \). \( \frac{5}{11} - \frac{27}{110} = \frac{3}{110} \). Thus the second digit is the integral part of the solution to \( 11b = 180 \). So it is 16. Thus \( \frac{5}{11} = ; 27, 16, \ldots \).

2. Recall that the algorithm says subtract the smaller from the larger until the smaller divides the larger. For 210, 231 the procedure starts with 210, 21 and 21 divides 210. Thus the greatest common divisor is 21. For the next we use the variant which divides the smaller into the larger and uses the remainder. Thus in the case of 308, 105 we start with 308 = 2(105) + 98, 105 = 98 + 7, 98 = 7(14) so the greatest common divisor is 7.

3. \( b = xy = (\frac{a}{2} + z)(\frac{a}{2} - z) = \frac{a^2}{4} - z^2 \). 

Hence (assuming \( \frac{a^2}{4} - b > 0 \) which we must) \( z = \sqrt{\frac{a^2}{4} - b} \). Now substitute for \( z \). For the second type setting \( y = z - 2 \) then \( a = x + y = x + z - 2 \). So \( x + z = a + 2 \).

\( b = xy + x - y = x(z - 2) + x - y = xz - 2x + x - y = xz + y - y = xz - a \).

Thus \( xz = a + b \). We now look at the problem. Take \( x, y \) the length and width respectively. The area is \( xy \) the excess of the length and the width is \( x - y \). The sum of the length and the width is \( x + y \). We therefore have \( x + y = 24 \), \( xy + x - y = 120 \). So if \( z = y - 2 \) then the first part of the procedure says that we are solving \( x + z = 26 \), \( xz = 144 \).

Thus \( x = \frac{26}{2} + \sqrt{13^2 - 144} = 13 + \sqrt{169 - 144} = 13 + 5 = 18 \). \( y = 24 - 18 = 6 \).

4. \( F_1^2 = 1 \) and \( F_1F_2 - F_1F_0 = 1 \cdot 2 - 1 \cdot 1 = 1 \). \( F_n = F_{n+1} - F_{n-1} \).

So \( F_n^2 = F_n(F_{n+1} - F_{n-1}) \). We now show that \( F_n^2 + \ldots + F_n^2 = F_nF_{n+1} \) by induction on \( n \). If \( n = 0 \) then \( F_0 = F_1 = 1 \) so the equation says 1 = 1. Assume that the formula is true for \( n \). We note that \( F_{n+1}^2 = F_nF_{n+2} - F_{n+1}F_n \).

So that \( F_nF_{n+2} = F_{n+1}F_n + F_n^2 \). Now using the assertion for \( n \) we have \( F_{n+1}F_{n+2} = (F_0^2 + \ldots + F_n^2) + F_n^2 \).

5. For this problem we look at the figure:
and consider the triangle $ABD$. Then height of the triangle with $AD$ thought of as the base is at most $AB$ since $AB$ would be the hypotenuse of a right triangle with one side the height. Thus since the area is the base times the height divided by 2 we have the area of the triangle $ABD$ is at most $\frac{ad}{2}$. This argument works for all of the triangles. The total area can be computed in two ways. The first as the area of $ABD$ plus that of $CBD$ and the second as the area of $ADC$ plus that of $ACB$. Computing in the first way we have the area is less than or equal to $\frac{ad}{2} + \frac{bc}{2}$ and in the second way it is less than or equal to $\frac{cd}{2} + \frac{ab}{2}$. We therefore see that the total area is less than or equal to the average of these two numbers

$$\frac{1}{2} \left( \frac{ad}{2} + \frac{bc}{2} \right) + \frac{1}{2} \left( \frac{cd}{2} + \frac{ab}{2} \right) = \frac{ad}{4} + \frac{bc}{4} + \frac{cd}{4} + \frac{ab}{4}.$$ 

if we multiply out we see that

$$\frac{(b + d)(a + c)}{4} = \frac{ad}{4} + \frac{bc}{4} + \frac{cd}{4} + \frac{ab}{4}.$$ 

So the Babylonian formula is indeed an upper bound. If it is exactly the area then all of the inequalities observed have to be equalities. This implies that every possible overestimate of a height was actually correct so every angle indicated must be a right angle. This implies that the Babylonian formula is correct for rectangles and no other type of figure.

6.a) $16 = 10 \cdot 1 + 6$. Assume that $16^k = 10 \cdot m + 6$. Then

$$16^{k+1} = 16(10 \cdot m + 6) = 10 \cdot (16m) + 10 \cdot 6 + 6 \cdot 6 = 10(16m + 6 + 3) + 6.$$ 

b) If $p$ is a prime and $p > 2$ then $p$ is odd. So $p = 4k + 1$ or $4k + 3$. If $n$ is perfect then $n = (2^p - 1)2^{p-1}$ with $p$ a prime. Thus

$$n = 2^{2p-1} - 2^{p-1}.$$ 

If $p = 4k + 1$ then $n = 2^{8k+1} - 2^{4k} = 2 \cdot 16^{2k} - 16^k = 2(10a + 6) - (10b + 6) = 6 + 10(a - b)$. Thus if $p = 4k + 1$ the last decimal digit of $n$ is 6. If $p = 4k + 3$ then

$$n = 2^{4(2k+1)+1} - 2^{4k+2} = 2 \cdot 16^{2k+1} - 4 \cdot 16^k = 2(10a + 6) - 4(10b + 6) = -12 + 20a - 40b = 8 + 20(a - 1) - 40b = 8 + 10c$$

with $c = 2(a - 1) - 4b$. Since $n > 0$, $c > 0$. So the last digit is 8. (This theorem is due to Gauss.)

7. $x = \frac{u}{v}$ in lowest terms. Thus $0 = \left( \frac{u}{v} \right)^3 + b \left( \frac{u}{v} \right)^2 + c \left( \frac{u}{v} \right) + d$ implies (after multiplying through by $v^3$) that $u^3 = v(-bu^2 - cuv - d)$. This implies that every prime factor of $v$ divides $u$. Since $u, v$ are relatively prime this implies that $v = 1$. So $x = u$. We now write the equation as $u(-u^2 - bu - c) = d$ so $u$ divides
Now consider the equation in question: $x^3 + 2x^2 - 13x + 10$. The divisors of 10 are 2 and 5. Plugging in 2 we see that $2^3 + 2 \times 2^2 - 13 \times 2 + 10 = 8 + 8 - 26 + 10 = 0$. Thus 2 is a root. Dividing $x^3 + 2x^2 - 13x + 10$ by $x - 2$ yields $x^2 + 4x - 5$. This factors as $(x - 1)(x + 5)$. Thus the roots are 1, 2, -5. Since all of the roots are real and distinct the term in the square root in Cardano’s formula would be negative. Thus a real number would be expressed in terms of two complex numbers. To apply Descartes’ rule of signs we have the signs of the coefficients follow the pattern $+ \rightarrow + \rightarrow - \rightarrow +$. So there should be 2 true roots ($+ \rightarrow -, - \rightarrow +$) and one false ($+ \rightarrow +$) which is correct.

8. The width is $\frac{1}{15} + \frac{1}{15} = \frac{2}{15}$ of the length. Since $\frac{2}{15} + 3 \cdot \frac{2}{15} = 1$, it follows that $40 + 3 \cdot 20 = 100$ is the area of the square whose side is the length. Thus the length is 10 and the width is 4.

9. a) This says $\frac{10x}{x+6} = \frac{40}{x+6}$. Hence $x = 2$.

b) The equations are $x + y = 10$ and $\frac{50}{x} \cdot \frac{40}{y} = 125$. Thus $x + y = 10$, $xy = 16$. So using the first part of problem 2 we see that

$$x = 5 + \sqrt{25 - 16} = 5 + 3 = 8.$$  
$$y = 5 - 3 = 2.$$  

10. We will refer to the picture

The angles $DBE$ and $DEF$ are right angles. The angles $BDE$ and $FDE$ are the same and the sides $BD$ and $DE$ are equal. Thus angle side angle implies that the triangles $DBC$ and $DEF$ are congruent. Hence $DC = FD$. We note that the Pythagorean theorem implies that $(BD)^2 + (BC)^2 = (DC)^2$. Now $BD = DA = \frac{1}{2}BA$, $BC = BA$, and $FD = DC$. Now

$$(FD)^2 = \frac{(BC)^2}{4} + (BC)^2 = \frac{5}{4}(BC)^2 = \frac{5}{4}(AB)^2.$$  

Hence $FD = \frac{\sqrt{5}}{2}AB$. So, $\frac{FA}{AB} = \frac{FD + BD}{AB} = \frac{FD + AB}{AB} = (\frac{\sqrt{5}}{2} + \frac{1}{2})\frac{AB}{AB}$. Thus $\frac{FA}{AB}$ is the Golden section and this implies that $\frac{FA}{AB} = \frac{\sqrt{5}}{2} - \frac{1}{2}$. The Golden ratio.

If we set $a = AB$ and $c = BF$ then we know that $(\frac{a+c}{a})^2 = \frac{a+c}{a} + 1$ that is $(a + c)^2 = a(a + c) + a$. So $a^2 + 2ac + c^2 = a^2 + ac + a^2$. Hence $ac + c^2 = a^2$. Dividing through by $c^2$ yields $(\frac{a}{c}) + 1 = (\frac{a}{c})^2$. So $\frac{a}{c}$ is the Golden section.