Solutions for the practice midterm Math 21C

1. We may look at this curve as a polar plot with \( r = \cos(4\theta) \). The method we used to do this was to first plot the function \( \cos(4x) \).

The part of interest to us is from 0 to \( 2\pi \). From 0 to \( \frac{\pi}{4} \), the graph goes from (1, 0) to (0, 0) above the x axis (on your paper you should indicate this with arrows) as we did in class. From \( \frac{\pi}{4} \) to \( \frac{\pi}{2} \) the value of \( r \) is negative. Thus it traces part of the curve in the negative \( x, y \) quadrant. Here is the graph.

(b) We have

\[
\frac{df}{dt} = -4 \sin(4t) \cos(t) - \cos(4t) \sin(t)
\]

\[
\frac{dg}{dt} = -4 \sin(4t) \sin(t) + \cos(4t) \cos(t).
\]

This implies that

\[
\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 = 16 \sin(4t)^2 \cos(t)^2 + 8 \sin(4t) \cos(t) \cos(4t) \sin(t) + \cos(4t)^2 \sin(t)^2 + 16 \sin(4t)^2 \sin(t)^2 - 8 \sin(4t) \sin(t) \cos(4t) \cos(t) + \cos(4t)^2 \cos(t)^2.
\]
We note that the middle terms cancel and using the fact that \(\cos(\theta)^2 + \sin(\theta)^2 = 1\) we have

\[
\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 = 16\sin(4t)^2 + \cos(4t)^2 = 15\sin(4t)^2 + 1.
\]

The arc length is therefore given by

\[
\int_0^{2\pi} \sqrt{15\sin(4t)^2 + 1} dt.
\]

(c) From the picture if \(t = \frac{\pi}{2}\) then \(f(t) = \cos(\pi) \cos\left(\frac{\pi}{2}\right) = -\frac{1}{\sqrt{2}}\) similarly we have \(g(t) = -\frac{1}{\sqrt{2}}\). Now \((\frac{df}{dt}(\frac{\pi}{2}), \frac{dg}{dt}(\frac{\pi}{2})) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\). Thus the line is given parametrically by

\[
x = \frac{-1 + t}{\sqrt{2}}, y = \frac{-1 - t}{\sqrt{2}}.
\]

2. a) \(\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\left(\frac{\pi}{4}\right) = 3 \times 7 \times \frac{1}{\sqrt{2}} = -\frac{21}{\sqrt{2}}\).

\(\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\left(\frac{\pi}{4}\right) = 3 \times 7 \times \frac{1}{\sqrt{2}} = \frac{21}{\sqrt{2}}\)

b) \(\mathbf{a} \cdot \mathbf{b} = 1 \cdot 2 + 3 \cdot (-1) + 1 \cdot 1 = 0\) and

\[
\mathbf{a} \times \mathbf{b} = \begin{vmatrix}
i & j & k \\
1 & 3 & 1 \\
2 & -1 & 1 \\
\end{vmatrix} = 4i - (-1)j - 6k = 4i + j - 6k.
\]

3. A picture of the triangle is

![Triangle Diagram]

The area of the parallelogram with adjacent sides \(\overrightarrow{PQ}\) and \(\overrightarrow{PR}\) is \(|\overrightarrow{PQ} \times \overrightarrow{PR}|\) after we think of \((x, y, z)\) as \((x, y, 0)\). Thus the area of the given triangle is \(1/2\) of that. We are thus looking at \(\frac{1}{2}|(2, 3, 0) \times (1, 2, 0)| = \frac{1}{2}||k|| = \frac{1}{2}||k|| = \frac{1}{2}||k|| = \frac{1}{2}.

2
4. The line is given by \( (x, y, z) = (1, 1, -1) + t(2, 0, 1) \). Thus \( x = 1 + 2t, y = 1, z = -1 + t \) give parametric equations for the line. Every pair of vectors on this line have displacement vector a multiple of \((2, 0, 1)\). The displacement vector \( \overrightarrow{PQ} = (2, -2, 3) \). Thus a normal vector for the plane is

\[
\begin{vmatrix}
i & j & k \\
2 & 0 & 1 \\
2 & -2 & 3 \\
\end{vmatrix} = 2i - 4j - 4k.
\]

Thus an equation for the plane is \( 2x - 4y - 4z = 2 \).

5. The curves \( c = k \) are the circles \( x^2 + y^2 = k + 2 \). We note that \( z \geq 2 \). The vertical traces are parabolas. Thus the surface looks like: