Material on group rings

Let $R$ be a ring and let $G$ be a group. If $f : G \to R$ is a function then we set $\text{supp}(f) = \{g \in G | f(g) \neq 0\}$. We set $RG$ equal to the set of all functions from $G$ to $R$ such that the set $\text{supp}(f)$ is finite. We define addition on $RG$ as follows: If $f_1, f_2 \in RG$ then we define $f_1 + f_2$ by

$$(f_1 + f_2)(g) = f_1(g) + f_2(g)$$

for $g \in G$. We note that since $0 + 0 = 0$ it follows that $\text{supp}(f_1 + f_2) \subset \text{supp}(f_1) \cup \text{supp}(f_2)$. We define multiplication as follows: if $f_1, f_2 \in RG$ then we define $f_1 \ast f_2$ by

$$(f_1 \ast f_2)(g) = \sum_{h \in \text{supp}(f_1)} f_1(h) f_2(h^{-1} g).$$

We note that if $f_1 \ast f_2(g) \neq 0$ then there exists $h \in \text{supp}(f_1)$ such that $h^{-1} g \in \text{supp}(f_2)$. Thus $g \in \{xy | x \in \text{supp}(f_1), y \in \text{supp}(f_2)\}$. Hence the cardinality (number of elements of) $\text{supp}(f_1 \ast f_2)$ is at most $|\text{supp}(f_1)||\text{supp}(f_2)| < \infty$.

We define 0 to be the function defined by 0($g$) = 0 for all $g \in G$ and if $f \in RG$ then $(-f)(g) = -f(g)$. Then $f + (-f) = 0$. Since

$$(f_1 + f_2 + f_3)(g) = f_1(g) + f_2(g) + f_3(g) = (f_1 + (f_2 + f_3))(g)$$

by the associative rule for addition in $R$ and

$$(f_1 + f_2)(g) = f_1(g) + f_2(g) = f_2(g) + f_1(g) = (f_2 + f_1)(g)$$

by the commutative rule for addition in $R$ we see that $RG$ is an abelian group under addition. We will now show that the multiplication on $RG$ is associative. Let $g \in G$ and $f_1, f_2, f_3 \in RG$ then

$$f_1 \ast (f_2 \ast f_3)(g) = \sum_{h \in \text{supp}(f_1)} f_1(h) (f_2 \ast f_3(h^{-1} g)) =$$

$$\sum_{h \in \text{supp}(f_1)} f_1(h) \left( \sum_{u \in \text{supp}(f_2)} f_2(u) f_3(u^{-1} h^{-1} g) \right).$$

We now use the associative and distributive rules for $R$ to rewrite this as

$$f_1 \ast (f_2 \ast f_3)(g) = \sum_{h \in \text{supp}(f_1)} \sum_{u \in \text{supp}(f_2)} f_1(h) f_2(u) f_3(u^{-1} h^{-1} g).$$

We now observe that the expression on the right hand of the equation can be written in the form

$$a \in \text{supp}(f_1), \quad b \in \text{supp}(f_2), \quad c \in \text{supp}(f_3) \quad f_1(a) f_2(b) f_3(c). \quad (*)$$

To see this we note that if $a = h, b = u, c = u^{-1} h^{-1} g$ then $abc = g$. Now assume that $a,b,c \in G$ and $abc = g$. Then set $h = a, u = b$. Then $huc = g$. So $g = u^{-1} h^{-1} g$. This proves the expression. Essentially the same argument proves that $(f_1 \ast (f_2 \ast f_3))(g)$ is equal to formula $(*)$. This proves that the multiplication
is associative. Finally, we check the distributive rules. Let \( f_1, f_2, f_3 \in RG \) and let \( g \in G \) then
\[
(f_1 * (f_2 + f_3))(g) = \sum_{h \in \text{supp}(f_1)} f_1(h)(f_2(h^{-1}g) + f_3(h^{-1}g)) = \\
\sum_{h \in \text{supp}(f_1)} f_1(h)f_2(h^{-1}g) + \sum_{h \in \text{supp}(f_1)} f_1(h)f_3(h^{-1}g) = (f_1 * f_2 + f_1 * f_3)(g).
\]
This proves one of the distributive laws the proof of the second is essentially the same.

Example. Let \( R \) be a ring and let \( G = \mathbb{Z}_2(= J_2) = \{[0],[1]\} \). If \( a, b \in RG \) then define \( f_{a,b} \) by \( f_{a,b}([0]) = a \) and \( f_{a,b}([1]) = b \). If \( f \in RG \) then \( f = f([0]), f([1]) \). We note that \( f_{a,b} + f_{c,d} = f_{a+c,b+d} \). We will now calculate the multiplication. We have
\[
(f_{a,b} * f_{c,d})([0]) = f_{a,b}([0])f_{c,d}([0]) + f_{a,b}([1])f_{c,d}([1]) = ac + bd.
\]
\[
(f_{a,b} * f_{c,d})([1]) = f_{a,b}([0])f_{c,d}([1]) + f_{a,b}([1])f_{c,d}([0]) = ad + bc.
\]
Thus \( f_{a,b} * f_{c,d} = f_{ac+bd,ad+bc} \).

**Exercises.**

1. Let \( G = \mathbb{Z}_3(= J_3) = \{[0],[1],[2]\} \). Let \( R \) be a ring. If \( a, b, c \in R \) then define \( f_{a,b,c} \) by \( f_{a,b,c}([0]) = a, f_{a,b,c}([1]) = b, f_{a,b,c}([2]) = c \). Calculate \( f_{a,b,c} + f_{a,v,w} \) and \( f_{a,b,c} * f_{a,v,w} \).

2. Let \( G = GL(2,\mathbb{R}) \) and let \( R = \mathbb{R} \). If \( f \in RG \) define \( \phi(f) = \sum_{g \in \text{supp}(f)} f(g)g \). Show that \( \phi \) is a ring homomorphism of \( RG \) onto \( M_2(\mathbb{R}) \).

3. Let \( R = \mathbb{R}, G = \mathbb{Z}_2 \) and let \( f_{a,b} \) be as the example above. Define \( \phi : \mathbb{R} \times \mathbb{R} \to RG \) by \( \phi(a, b) = f_{\frac{a+b}{2}, \frac{a-b}{2}} \). Show that \( \phi \) is a ring isomorphism.