Some Practice Problems involving Green’s, Stokes’, Gauss’ theorems.

1. Let \( \mathbf{x}(t) = (a \cos t^2, b \sin t^2) \) with \( a, b > 0 \) for \( 0 \leq t \leq \sqrt{2\pi} \). Calculate \( \int_{\mathbf{x}} x \, dy \). Hint: \( \cos^2 t = \frac{1 + \cos 2t}{2} \).

2. Let \( \mathbf{F} = \frac{-yi + zj}{x^2 + y^2} \).
   a) Use Green’s theorem to explain why \( \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = 0 \) if \( \mathbf{x} \) is the boundary of a domain that doesn’t contain 0.
   b) Let \( \mathbf{x}(t) = (\cos t, 3 \sin t), 0 \leq t \leq 2\pi \) and \( \mathbf{F} = \frac{-yi + zj}{x^2 + y^2} \). Calculate \( \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} \). Hint: Consider the domain between \( \mathbf{x} \) and the circle \( \mathbf{y}(t) = (\cos t, \sin t) \). Use part a) to see that \( \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{y}} \mathbf{F} \cdot d\mathbf{s} \).

3. Which if the following vector fields is of the form \( \nabla f \)? If it is compute an \( f \).
   a) \( \mathbf{F} = x^2 \mathbf{i} + xy \mathbf{j} \).
   b) \( \mathbf{F} = x^2 \mathbf{i} - y^2 \mathbf{j} \).
   c) \( \mathbf{F} = yi - xj \).
   d) \( \mathbf{F} = (3x^2y + 2xy^2) \mathbf{i} + (x^3 + 2x^2y + 3y^2) \mathbf{j} \).

4. Let \( S \) be the surface \( z = 4 - x^2 - y^2, z \geq -3 \) and let \( \mathbf{F} = (2xyz + 3z) \mathbf{i} + x^2y \mathbf{j} + \cos(xyz) e^x \mathbf{k} \). Calculate
   \[ \int \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \]
   Hint: Observe that \( \partial S \) is the boundary of another surface.

5. Let \( S \) be the union of the surfaces \( z = x^2 + y^2 - 1 \) with \( z \leq 0 \) and \( x^2 + y^2 + z^2 = 1, z \geq 0 \). Let \( \mathbf{x}(t) = (\cos t, \sin t, 0), 0 \leq t \leq 2\pi \). Calculate \( \int \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \) for \( \mathbf{F} \) an arbitrary \( C^1 \) vector field using Stokes’ theorem. Do the same using Gauss’s theorem (that is the divergence theorem).

6. Let \( V \) be the solid cylinder \( x^2 + y^2 \leq 1, |z| \leq 1 \). Describe the boundary of \( V \). Orient the boundary using the outward normal and use Gauss’s theorem to calculate \( \int \int_{\partial V} \mathbf{F} \cdot d\mathbf{S} \) with \( \mathbf{F} = x \mathbf{k} + y \mathbf{j} + z \mathbf{i} \).