RLC Circuits


This problem is based on Boyce-DePrima, "Electric Circuits", Section 3.7 p. 202

An RLC circuit is a circuit composed of a *resistor* (of resistance $R$ Ohms), an *inductor* (of inductance $L$ Henrys), a *capacitor* (or capacitance $C$ Farads), and an *impressed voltage* (of $E(t)$ Volts) wired in series (see figure 1). The impressed voltage may be time dependent, whereas $R, L, C$ are constants.

![RLC Circuit](https://upload.wikimedia.org/wikipedia/commons/f/fb/RLC_series_circuit_v1.svg)

Let $Q(t)$ denote the charge on the capacitor at time $t$ and let $I(t)$ denote the current through the circuit at time $t$. Then we have the relation

$$I = \frac{dQ}{dt}. \quad (1)$$

The voltage drops across the circuit elements are

$$V_R = IR,$$

$$V_C = Q/C,$$

$$V_L = L\frac{dI}{dt}.$$
Kirchhoff’s Voltage Law states that the impressed voltage equals the sum of the voltage drops. That is

\[ E(t) = IR + Q/C + L \frac{dI}{dt}. \]

1. Show that this leads the differential equation

\[ LQ'' + RQ' + \frac{1}{C}Q = E(t) \] (2)

for the charge on the capacitor.

2. Undamped free oscillations. Suppose that there is no resistance in the circuit \((R = 0)\) and no impressed voltage \(E(t) = 0\).

(a) Both an inductor and a capacitor store and release energy. The electrical energy stored in the capacitor is

\[ U_C(t) = \frac{Q(t)^2}{2C}, \]

and the energy stored in the inductor is

\[ U_L(t) = \frac{LI(t)^2}{2}. \]

The total electrical energy stored in the circuit is thus

\[ U(t) = \frac{1}{2} \left( \frac{Q(t)^2}{C} + LI(t)^2 \right). \]

Without solving equation (2), show that energy is conserved, i.e. \(U(t)\) is constant (compare problem 30 in section 3.7 of Boyce-DePrima).

Hint: How do you tell if a function is constant? Now use equations (2) and (1).

(b) Show that the charge is

\[ Q(t) = A \cos(\omega_0 t - \delta) \]

where \(A\) and \(\delta\) are arbitrary constants and

\[ \omega_0 = \frac{1}{\sqrt{LC}}. \]

We call \(A\) the amplitude of oscillation, \(\delta\) the phase and \(\omega_0\) the natural, or resonant frequency.
(c) Show that the current is
\[ I(t) = -A\omega_0 \sin(\omega_0 t - \delta). \]

(d) Using the expressions for \( Q \) and \( I \) from parts (b) and (c), verify explicitly that
\[ U(t) = \frac{A^2}{2C} \]
is constant.

3. Underdamped free oscillations. Suppose \( R < 2\sqrt{L/C} \) and \( E(t) \) is a given function.

(a) Show that
\[ \frac{dU}{dt} = Q'(t)[E(t) - RQ'(t)] \]
and thus in the presence of resistance or an impressed voltage, energy is generally not conserved.

(b) Show that the general solution of the homogeneous equation \( (E(t) = 0) \) \[Q(t) = \exp \left(-\frac{R}{2L}t\right) A\cos(\mu t - \delta)\]
where the amplitude \( A \) and the phase \( \delta \) are arbitrary constants, and
\[ \mu = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \]
is the quasi-frequency. Thus if \( R < 2\sqrt{L/C} \) the circuit undergoes a damped oscillation.

(c) Let
\[ E(t) = -R\omega_0 \sin(\omega_0 t), \]
where \( \omega_0 = 1/\sqrt{LC} \) is the resonant frequency. Show by direct substitution into (2), that a particular solution of the non-homogeneous equation \[Q(t) = \cos(\omega_0 t).\]

Using part (a) show that energy is conserved for this particular solution. Hence, for this particular solution, the loss of energy via heat in the resistor is exactly compensated for by the addition of energy from the impressed voltage.