Math 280C, Spring 2005

Homework 1

Due April 6


2. Exercise 54, p. 440.

3. Let \( \{\xi_1, \xi_2, \ldots\} \) be an iid sequence, and let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a symmetric (measurable) function of two variables such that \( \mathbb{E}|f(\xi_1, \xi_2)| < \infty \). Define

\[
U_n := \left( \frac{n}{2} \right)^{-1} \sum_{1 \leq j < k \leq n} f(\xi_j, \xi_k), \quad n \geq 2,
\]

and

\[
\mathcal{G}_n := \sigma\{U_n, U_{n+1}, U_{n+2}, \ldots\}, \quad \mathcal{G}_\infty := \cap_n \mathcal{G}_n.
\]

(a) Show that \( \{U_n, \mathcal{G}_n\}_{n \geq 2} \) is a reverse martingale.
(b) Deduce that \( U_\infty := \lim_{n \to \infty} U_n \) exists, both a.s. and in \( L^1 \).
(c) Define \( \mathcal{H}_m := \sigma\{\xi_m, \xi_{m+1}, \ldots\} \) and \( \mathcal{H}_\infty := \cap_m \mathcal{H}_m \), so that \( \mathcal{H}_\infty \) is the tail \( \sigma \)-field of \( \{\xi_n\} \). Show that \( U_\infty \) is equal a.s. to an \( \mathcal{H}_\infty \)-measurable random variable. Deduce from this that \( U_\infty = \mathbb{E}[f(\xi_1, \xi_2)] \) a.s.

4. Let \( \{X_n\} \) be a Markov chain with countable state space \( S \) and one-step transition probabilities \( p(x, y) = P_x(X_1 = y) \). Let \( P = (p(x, y))_{x,y \in S} \) be the corresponding transition matrix, and let \( P^k \) denote the \( k \)-th power of \( P \); this is the \( k \)-step transition matrix with entries given by

\[
p^k(x, y) = P_x[X_k = y] = P[X_{k+m} = y|X_m = x].
\]

Let \( f : S \to \mathbb{R} \) be a bounded function and define \( P^kf(x) := \sum_{y \in S} p^k(x, y) f(y) \). Show that if \( N \) is a fixed positive integer, then \( \{P^{N-n}f(X_n)\}_{0 \leq n \leq N} \) is a martingale.