Math 280C, Spring 2005

Homework 2 — Due April 18

In the problems below, $X = (X_n)_{n=0}^\infty$ is a Markov chain with countable state space $S$ and transition probability matrix $p$. We suppose that $X$ has been constructed on the sequence space $\Omega = S^{\{0,1,2,\ldots\}}$, and that $P_x$ is the probability measure on $\Omega$ corresponding to the initial condition $X_0 = x$. Other notation is that used in class.

Problems marked with an * should be attempted, but need not be handed in.

1. Let $f : S \to [0,\infty)$, and define
   
   $$u(x) := \mathbb{E}_x \left[ \sum_{k=0}^\infty f(X_k) \right], \quad x \in S.$$  

   Notice that $u : S \to [0,\infty]$. Fix $x_0 \in S$ and suppose that $u(x_0) < \infty$.
   
   (a) Show that $\sum_{y \in S} p(x,y) u(y) \leq u(x)$ for all $x \in S$.
   
   (b) Show that $\mathbb{E}_{x_0}[u(X_n)] < \infty$ for all $n = 0,1,2,\ldots$.
   
   (c) Deduce from (a) and (b) that $u(X_n)$, $n \geq 0$, is a non-negative supermartingale with respect to $P_{x_0}$.
   
   (d) Show that $\lim_{n \to \infty} u(X_n) = 0$, both $P_{x_0}$-a.s. and in $L^1(P_{x_0})$.

2. Let $Z$ be a bounded $\mathcal{F}_\infty$-measurable function on $\Omega$ with the “shift invariance” property $Z(\theta_n \omega) = Z(\omega)$ for all $\omega \in \Omega$ and all $n = 1,2,\ldots$. Define a bounded function $h : S \to \mathbb{R}$ by the formula $h(x) := \mathbb{E}_x[Z]$, $x \in S$.

   (a) Show that $h(X_n)$, $n \geq 0$, is a $P_{\mu}$-martingale for any initial distribution $\mu$.

   (b) Show that $\lim_{n \to \infty} h(X_n) = Z$, both $P_{\mu}$-a.s. and in $L^1(P_{\mu})$.

   [Hint: Compute $\mathbb{E}_\mu[Z|\mathcal{F}_n]$ by using the shift invariance property of $Z$.]

*3. Fix $B \subset S$ and define a stopping time $D := \min\{n \geq 0 : X_n \in B\}$. Suppose that $S \setminus B$ is a finite set and that $P_x[D < \infty] > 0$ for all $x \in S \setminus B$. [Of course, $P_x[D = 0] = 1$ if $x \in B$. Also, if $D(\omega) \geq n$ then $D(\omega) = n + D(\theta_n \omega)$.

   (a) Show that $P_x[D < \infty] = 1$ for all $x \in S$. [Hints: The indicator $1_{\{D<\infty\}}$ takes on only the values 0 and 1, and $1_{\{D<\infty\}} = \lim_{n \to \infty} P_{\mu}[D < \infty|\mathcal{F}_n]$ almost surely. But $\{D < \infty\} \supset \theta_n^{-1}\{D < \infty\}$, and $P_{\mu}[\theta_n^{-1}\{D < \infty\}|\mathcal{F}_n] = P_{X_0}[D < \infty]$. Because $S \setminus B$ is a finite set, the lower bound $\delta := \inf\{P_x[D < \infty] : x \in S\}$ is strictly positive.]

   (b) Show that there is a positive integer $N$ such that $P_x[D \geq kN] \leq 2^{-k}$, $k = 0,1,2,\ldots$. [Hint: By (a), $\lim_{N \to \infty} P_x[D \geq N] = 0$ for all $x \in S$. Thus, because $S \setminus B$ is finite, there exists $N \in \mathbb{N}$ such that $P_x[D \geq N] \leq 1/2$ for all $x \notin B$. This inequality
holds trivially for $x \in B$, because $P_x[D = 0] = 1$ for $x \in B$. Now $P_x[D \geq 2N] = P_x[D \geq
\text{etc.}]

(c) Use part (b) to show that $E_x[D] < \infty$ for all $x \in S$. Deduce that $x \mapsto E_x[D]$ is a bounded function. [Hint: Use the formula $E_x[D] = \sum_{n=1}^{\infty} P_x[D \geq n]$, bearing in mind that $P_x[D \geq kN + j] \leq P_x[D \geq kN]$ for $j = 1, 2, \ldots, N - 1, k = 0, 1, 2, \ldots].$

4. Fix $B \subset S$ and define a stopping time $D := \min\{n \geq 0 : X_n \in B\}$. Suppose that $S \setminus B$ is a finite set and that $P_x[D < \infty] > 0$ for all $x \in S \setminus B$.

(a) Show that if $g(x) := E_x[D]$ (a bounded function, by problem 3) then

$$g(x) = 1 + \sum_{y \in S} p(x, y)g(y), \quad \forall x \in S \setminus B.$$  

(b) Show that if $h$ is any bounded function satisfying

$$(\dagger) \quad h(x) = 1 + \sum_{y \in S} p(x, y)h(y), \quad \forall x \in S \setminus B,$$

then

$$Y_n := h(X_n \wedge D) + (n \wedge D), \quad n \geq 0,$$

is a $P_x$-martingale for each $x \in S$.

(c) Show that if $h$ is a bounded function satisfying $(\dagger)$ and if $h(x) = 0$ for all $x \in B$, then $h(x) = E_x[D]$ for all $x \in S$. [Hint: Define $f(x) := h(x) - E_x[D]$ and (using (a) and (b)) show that $f(X_n \wedge D)$ is a bounded martingale. Now compute $\lim_{n \to \infty} h(X_n \wedge D)$, making use of the fact that $X_D \in B$ on $\{D < \infty\}$.]

5. In this problem, assume that $X$ is irreducible. Let $v : S \to [0, \infty)$ be such that $E_x[v(X_1)] \leq v(x)$ for all $x$ in the complement of some finite set $F \subset S$. Suppose also that (i) $v(x) > 0$ for all $x \in F$, but (ii) $\inf\{v(x) : x \in S\} = 0$. Show that $X$ is transient. [Hint: Define $\gamma := \min\{v(x) : x \in F\}$ and notice that $\gamma > 0$. Argue by contradiction, showing that if $X$ has a recurrent point, then $v(x) \geq \delta$ for all $x \in E$, violating hypothesis (ii).]