1. Let \( X(t), t \geq 0 \), be a general birth-and-death process with state space \( \{0, 1, 2, \ldots\} \), and strictly positive birth rates \( \lambda_0, \lambda_1, \ldots \) and death rates \( \mu_1, \mu_2, \ldots \). Let \( T \) be the first time \( t \geq 0 \) that \( X(t) \) is equal to 0 or 10, and define
\[
\lambda_i = P(X(T) = 0|X(0) = i), \quad i = 0, 1, 2, \ldots, 10.
\]
Show that the \( \lambda_i \) satisfy the conditions
(i) \( \lambda_0 = 1, \lambda_{10} = 0 \).
(ii) \( (\lambda_i + \mu_i)u_i = \lambda_i u_{i+1} + \mu_i u_{i-1}, \ i = 1, 2, \ldots, 9 \).
Do not attempt to solve these equations.

2. \( X(t), t \geq 0 \), is a birth-and-death process with state space \( \{0, 1\} \) and transition probability matrix
\[
P(t) = \begin{pmatrix}
2/3 + (1/3)e^{-3t} & 1/3 - (1/3)e^{-3t} \\
2/3 - (2/3)e^{-3t} & 1/3 + (2/3)e^{-3t}
\end{pmatrix}.
\]
(a) Verify that \( P(t) \) satisfies the Chapman-Kolmogorov equation: \( P(t+s) = P(t)P(s) \), for all \( t, s \geq 0 \).
(b) Find the birth rate \( \lambda_0 \) and the death rate \( \mu_1 \) for \( X \).

3. Birds are perched along a wire (= \([0, \infty)\)) in accordance with a Poisson process of rate \( \lambda \) per unit distance. For \( t \geq 0 \), let \( D(t) \) be the (random) distance from \( t \) to the nearest bird.
(a) Show that
\[
P[D(t) > x] = \begin{cases}
e^{-2\lambda x} & \text{if } 0 < x < t; \\
e^{-\lambda(x+t)} & \text{if } x > t.
\end{cases}
\]
(b) Find \( E[D(t)] \).

[Hint: \( D(t) > x \) if and only if there are no birds in the interval \([\max(t-x,0), t+x]\).]

4. Let \( X(t) \) be a Markov chain with state space \( \{1, 2, \ldots, N\} \) and transition matrix \( P(t) = [P_{ij}(t)]_{i,j=1}^{N} \). Assume that \( P(t) \) is irreducible in the sense that \( P_{ij}(t) > 0 \) for all \( i,j \in \{1,2,\ldots,N\} \) and all \( t > 0 \). Under these conditions, the limit probabilities
\[\pi_j = \lim_{t \to \infty} P_{ij}(t)\]
exist, and \( \pi = (\pi_1, \ldots, \pi_N) \) is the unique probability distribution on the state space \( \{1, \ldots, N\} \) such that \( \pi A = 0 \), where \( A = P'(0) \). Show that the following three properties are equivalent:
(a) \( \pi_j = 1/N, \ j = 1, 2, \ldots, N \). (That is, \( \pi \) is the uniform distribution.)
(b) \( \sum_{i=1}^{N} a_{ij} = 0 \) for all \( j = 1, 2, \ldots, N \). (That is, the \textit{column} sums of \( A \) are all 0.)
(c) \( \sum_{i=1}^{N} P_{ij}(t) = 1 \) for all \( j = 1, 2, \ldots, N \) and all \( t > 0 \). (That is, the \textit{column} sums of \( P(t) \) are all 1, for every \( t > 0 \).)

5. Let \( N(t), t \geq 0 \), be a renewal counting process whose interarrival times \( X_1, X_2, \ldots \) have the common continuous-type density function \( f \). Thus \( N(t) = \# \{k \geq 1 : W_k \leq t\} \), where \( W_k = X_1 + \cdots + X_k \) for \( k = 1, 2, \ldots \). We now “thin” the renewal process as follows. Let \( Z_1, Z_2, \ldots \), be a sequence of independent Bernoulli random variables, independent of \( N \), with \( P[Z_k = 0] = P[Z_k = 1] = 1/2, k = 1, 2, \ldots, \) and define a new counting process \( \hat{N}(t) = \# \{k \geq 1 : W_k \leq t \text{ and } Z_k = 1\} \).
(a) Show that \( \hat{N}(t), t \geq 0 \) is a renewal counting process and that the probability density function \( g \) for its interarrival times is given by
\[
g(x) = \sum_{k=1}^{\infty} (1/2)^k f^{*k}(x), \quad x > 0,
\]
where \( f^{*k} = f \ast f \ast \cdots \ast f \) (k factors), the \( k \)-fold convolution of \( f \), is the density function of \( W_k \).
(b) Use the result of part (a) to find an explicit formula for \( g \) when \( N \) is a Poisson process.