1. (a) Let \( a, b > 0 \) and let a point of mass \( m \) move along the curve in \( \mathbb{R}^3 \) defined by \( \mathbf{c}(t) = (a \cos t, b \sin t, t^2) \). Describe the curve and find the tangent line at \( t = \pi \).

**Solution.** This is a *stretched elliptical helix*. The derivative is
\[
\mathbf{c}'(t) = (-a \sin t, b \cos t, 2t).
\]
The tangent line is
\[
\mathbf{l}(t) = \mathbf{c}(\pi) + t\mathbf{c}'(\pi)
= (-a, 0, \pi^2) + t(0, -b, 2\pi)
= (-a, -tb, \pi^2 + 2t\pi).
\]

(b) Find the force acting on the particle at \( t = \pi \).

**Solution.** The acceleration is
\[
\mathbf{c}''(t) = (-a \cos t, -b \sin t, 2),
\]
so \( \mathbf{c}''(\pi) = (a, 0, 2) \). Thus, the force is \( \mathbf{F} = m\mathbf{a} = m(a, 0, 2) \).
(c) Define the function \( E(z) \) by

\[
E(z) = \int_0^{\pi/2} \sqrt{1 - z^2 \sin^2 t} \, dt.
\]

Find a formula for the circumference of the ellipse

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,
\]

where \( 0 < a < b \), in terms of \( E(z) \).

**Solution.** The circumference of the ellipse is

\[
C = 4 \int_0^{\pi/2} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} \, dt.
\]

Now substitute \( \cos^2 t = 1 - \sin^2 t \), and pull out a factor of \( b \) to give the final desired formula:

\[
C = 4b E(\sqrt{1 - (a^2/b^2)}).
\]

(d) Let \( f : \mathbb{R}^3 \to \mathbb{R} \) be defined by

\[
f(x, y, z) = -z + \exp(x^2 - y^2).
\]

In what directions starting from \((1, 1, 1)\) is \( f \) decreasing at 50% of its maximum rate of change?

**Solution.** The gradient is

\[
\nabla f(x, y, z) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = (2x \exp(x^2 - y^2), -2y \exp(x^2 - y^2), -1).
\]

Thus, the direction of fastest decrease at \((1, 1, 1)\) is \(-\nabla f(1, 1, 1) = (-2, 2, 1)\). Suppose now that \( \mathbf{n} \) is the direction that \( f \) decreases at 50% of its maximum rate and \( \|\mathbf{n}\| = 1 \). Then, because the rate of change of \( f \) in the direction \( \mathbf{n} \) at the point \((1, 1, 1)\) is \( \mathbf{n} \cdot \nabla f(1, 1, 1) \), we get

\[
\mathbf{n} \cdot (-\nabla f(1, 1, 1)) = \frac{1}{2} \| -\nabla f(1, 1, 1) \|.
\]
Let the angle between $n$ and $-\nabla f(1,1,1)$ be denoted $\theta$ (taken to be the acute angle). Then from the preceding discussion,

$$\cos \theta \|n\| \cdot \| -\nabla f(1,1,1) \| = \frac{1}{2} \| -\nabla f(1,1,1) \|.$$ 

Thus, $\cos \theta = 1/2$, so $\theta = \pi/3$. Therefore, the directions are those that make an angle of $\pi/3$ with the vector $(-2,2,1)$; that is, a cone centered on the vector $(-2,2,1)$ with a base angle of $2\pi/3$.

(e) If $f$ is a smooth function of $(x,y), C$ is the circle $x^2 + y^2 = 1$ and $D$ is the unit disc $x^2 + y^2 \leq 1$, is it true that

$$\int_C \sin(xy) \frac{\partial f}{\partial x} \, dx + \sin(xy) \frac{\partial f}{\partial y} \, dy = \int\int_D \cos(xy) \left[ y \frac{\partial f}{\partial y} - x \frac{\partial f}{\partial x} \right] \, dx \, dy?$$

Solution. Yes, this is true. Let

$$P = \sin(xy) \frac{\partial f}{\partial x} \quad \text{and} \quad Q = \sin(xy) \frac{\partial f}{\partial y}$$

and apply Green’s theorem:

$$\int_C P \, dx + Q \, dy = \int\int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy.$$ 

We have

$$\frac{\partial Q}{\partial x} = y \cos xy \frac{\partial f}{\partial y} + \sin(xy) \frac{\partial^2 f}{\partial x \partial y}$$

and

$$\frac{\partial P}{\partial y} = x \cos xy \frac{\partial f}{\partial x} + \sin(xy) \frac{\partial^2 f}{\partial x \partial y}.$$ 

The terms containing second derivatives of $f$ cancel, so one gets the desired result.

2. (a) Find the extrema of the function

$$f(x,y,z) = \exp [z^2 - 2y^2 + x^2],$$

subject to the two constraints $g_1(x,y,z) = 1$ and $g_2(x,y,z) = 0$, where

$$g_1(x,y,z) = x^2 + y^2, \quad g_2(x,y,z) = z^2 - x^2 - y^2.$$
**Solution.** We present two possible solutions.

**Method 1.** The first method uses Lagrange multipliers. According to the Lagrange multiplier method, the extrema are among the solutions of the equations

\[
\begin{align*}
g_1(x, y, z) &= 1 \\
g_2(x, y, z) &= 0 \\
\nabla f &= \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2.
\end{align*}
\]

This gives the five equations

\[
\begin{align*}
x^2 + y^2 &= 1 \\
z^2 - x^2 - y^2 &= 0 \\
2xf &= 2x\lambda_1 - 2x\lambda_2 \\
-4yf &= 2y\lambda_1 - 2y\lambda_2 \\
2zf &= 2z\lambda_2.
\end{align*}
\]

Because \( f > 0 \) and, from the first two equations, \((x, y) \neq (0, 0)\), so \( z \neq 0 \). Thus, from the last equation, \( f = \lambda_2 \). There are two cases:

i. \( y \neq 0 \): Here \( \lambda_1 = -f \) from the fourth equation. Substituting into the third equation gives \( x = 0 \). Thus, \( y = \pm 1 \) and \( z = \pm 1 \).

ii. \( x \neq 0 \): Here from the third equation, \( \lambda_1 = 2f \). Substituting into the fourth equation gives \( y = 0 \). Thus, \( x = \pm 1 \) and \( z = \pm 1 \).

Evaluating \( f \) at each of the eight candidate gives a minimum of \( e^{-1} \) at the four points \((0, \pm 1, \pm 1)\) and a maximum of \( e^2 \) at the four points \((\pm 1, 0, \pm 1)\).

**Method 2.** The constraint set is the collection of two circles

\[
x^2 + y^2 = 1, z = 1 \quad \text{and} \quad x^2 + y^2 = 1, z = -1.
\]

Thus, we are extremizing the function \( h(x, y) = \exp[1 - 2y^2 + x^2] \) over the circle \( x^2 + y^2 = 1 \). That is, the function \( k(y) = \exp[2 - 3y^2] \) for \( y \) between \(-1\) and \( 1 \). Thus, because the function \( k(y) \) is decreasing in \( y^2 \), the maximum of \( e^2 \) occurs at \((\pm 1, 0, \pm 1)\) and the minimum of \( e^{-1} \) occurs at \((0, \pm 1, \pm 1)\).

\[ \blacklozenge \]

(b) Suppose \( S \) is the level surface in \( \mathbb{R}^n \) given by

\[
\{ x \in \mathbb{R}^n \mid g(x) = 0 \}
\]

for a smooth function \( g : \mathbb{R}^n \rightarrow \mathbb{R} \). Suppose \( \phi : \mathbb{R} \rightarrow \mathbb{R}^n \) is a curve whose image lies in \( S \); that is, for every \( x, \phi(x) \in S \). Show that \( \nabla g(\phi(0)) \) is perpendicular to \( \phi'(0) \).
Solution. The chain rule applied to $g(\phi(t)) = 0$ gives the equality $\nabla g(\phi(0)) \cdot \phi'(0) = 0$, which proves the assertion.

(c) Find the minimum distance between the circle $x^2 + y^2 = 1$ and the line $x + y = 4$.

Solution. We wish to minimize the square of the distance function from a point $(x, y)$ on the circle to a point $(u, v)$ on the line, namely, $f(x, y, u, v) = (x - u)^2 + (y - v)^2$, subject to the constraints

$$g_1(x, y, u, v) = x^2 + y^2 - 1 = 0$$
and
$$g_2(x, y, u, v) = u + v - 4 = 0.$$  

The Lagrange multiplier method gives

$$g_1 = 0$$
$$g_2 = 0$$
$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2,$$

which gives the following six equations

$$x^2 + y^2 = 1$$
$$u + v = 4$$
$$2(x - u) = 2\lambda_1 x$$
$$2(y - v) = 2\lambda_1 y$$
$$-2(x - u) = \lambda_2$$
$$-2(y - v) = \lambda_2.$$  

First, notice that $\lambda_1 \neq 0$, for then we would get $x = u, y = v$, which cannot happen since the line does not intersect the circle. Hence

$$x = \frac{x - u}{\lambda_1} = -\frac{1}{2} \frac{\lambda_2}{\lambda_1} = \frac{y - v}{\lambda_1} = y.$$  

Substituting $x = y$ into the last two of the preceding equations gives $u = v$ as well. From the first and second equations, we find that the potential extrema are located at $x = y = \pm \sqrt{2}/2$. Plugging these values into the distance function, we find the values

$$d_1 = \sqrt{\left(-\frac{\sqrt{2}}{2} - 2\right)^2 + \left(-\frac{\sqrt{2}}{2} - 2\right)^2} = 2\sqrt{2} + 1.$$
and
\[ d_2 = \sqrt{\left(\frac{\sqrt{2}}{2} - 2\right)^2 + \left(\frac{\sqrt{2}}{2} - 2\right)^2} = 2\sqrt{2} - 1. \]

Thus, the minimum distance is \( 2\sqrt{2} - 1 \). ♦

One can also do this problem geometrically by drawing the line and the circle and the line orthogonal to them, both to compute the intersection points and then calculate the distance between them. However, in this method, one has to be very careful to justify the method.

3. Evaluate the integral
\[ \int_0^1 \int_y^1 e^{-x^2/2} \, dx \, dy. \]

Drawing a figure and switching the order of integration:
\[ \int_0^1 \int_y^1 e^{-x^2/2} \, dx \, dy = \int_0^1 \int_0^x e^{-x^2/2} \, dy \, dx = \int_0^1 xe^{-x^2/2} \, dx = 1 - e^{-1/2}. \]

Solution. Evaluate
\[ \iiint_D x \, dx \, dy \, dz, \]
where \( D \) is the tetrahedron with vertices
\[ (0, 0, 0), (1, 0, 0), (0, 2, 0), \left(0, 0, \frac{1}{2}\right). \]

Solution. First compute or observe that the plane \( x + \frac{1}{2}y + 2z = 1 \) contains the tetrahedral face passing through the points
\[ (1, 0, 0), (0, 2, 0), \left(0, 0, \frac{1}{2}\right). \]

Thus, one gets
\[ \iiint_D x \, dx \, dy \, dz = \int_0^1 \int_0^{2-2x} \int_0^{1/2-1/2x-1/4y} x \, dz \, dy \, dx \]
\[ = \int_0^1 \int_0^{2-2x} \frac{x}{2} - \frac{x^2}{2} - \frac{xy}{4} \, dy \, dx \]
\[ = \int_0^1 \left( \frac{x}{2} - x^2 + \frac{x^3}{2} \right) \, dx \]
\[ = \frac{1}{24}. \] ♦
4. Determine whether each of the following statements below is True or False. If true, Justify (give a brief explanation or quote a relevant theorem from the course), and if false, give an explanation, or a Counterexample.

(a) For smooth functions \( f \) and \( g \),
\[
\text{div}(\nabla f \times \nabla g) = 0.
\]

**Solution.** This is one of the vector identities; it is true and follows by equality of mixed partials. In fact,
\[
\nabla f \times \nabla g = \begin{vmatrix}
    \mathbf{i} & \mathbf{j} & \mathbf{k} \\
    \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\
    \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z}
\end{vmatrix}
\]
\[
= \left( \frac{\partial f}{\partial y} \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial g}{\partial y} \right) \mathbf{i} - \left( \frac{\partial f}{\partial x} \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial g}{\partial x} \right) \mathbf{j} + \left( \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \right) \mathbf{k}.
\]

After taking the divergence, we see that all terms cancel (the student should be prepared to say which terms cancel with which terms). ♦

(b) There exist conservative vector fields \( F \) and \( G \), such that
\[
F \times G = (x, y, z).
\]

**Solution.** This is false. By part (a), the left side has zero divergence, but the right side clearly does not (its divergence is 3). ♦

(c) By a “symmetry” argument, the following holds
\[
\int_0^1 \int_{-2}^2 \int_1^2 z^{-1} x y e^{x^2} (y^2 + 1)^{-3} 2(y^6 + x^6 y^7 + 5) \, dy \, dx \, dz
\]
\[
= \int_{-\pi}^\pi \cos^4(t) \sin(t) \, dt
\]

**Solution.** This is true; both sides are zero by symmetry; the first is odd in the \( x \)-integral and the second is zero because it is an odd function of \( t \). ♦
(d) The surface integral of the vector field
\[ F(x, y, z) = (x + y^2 + z^2) \mathbf{i} + (x^2 + y + z^2) \mathbf{j} + (x^2 + y^2 + z) \mathbf{k} \]
over the unit sphere equals \(4\pi\).

**Solution.** This is true. By the divergence theorem, the flux is the integral of the divergence. But the divergence is the constant 3, so the integral over the unit ball is \(3 \times (4\pi)/3 = 4\pi\). ♦

(e) If \(F\) is a smooth vector field on \(\mathbb{R}^3\),
\[ S_1 = \{(x, y, 0) \mid x^2 + y^2 \leq 1\}, \]
oriented with the upward normal and the upper hemisphere,
\[ S_2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1, z \geq 0\}, \]
also oriented with the upward normal, then
\[ \int_{S_1} (\nabla \times F) \cdot dS = \int_{S_2} (\nabla \times F) \cdot dS. \]

**Solution.** This is true. By Stokes’ theorem, both sides equal the integral of \(F\) around the common boundary, the unit circle in the plane. One can also do this by Gauss’ theorem by writing the difference of the two sides as the surface integral over the boundary of the region enclosed, invoking the divergence theorem and using the fact that the divergence of a curl is zero. ♦

5. Let \(W\) be the region in \(\mathbb{R}^3\) defined by \(x^2 + y^2 + z^2 \leq 1\) and \(y \leq x\).

(a) Compute
\[ \iiint_W (1 - x^2 - y^2 - z^2) \, dx \, dy \, dz. \]

**Solution.** Consider spherical coordinates in \(\mathbb{R}^3\):
\[
\begin{align*}
x &= r \cos \theta \sin \phi \\
y &= r \sin \theta \sin \phi \\
z &= r \cos \phi,
\end{align*}
\]
with
\[
\begin{align*}
r &\in [0, \infty), \\
\phi &\in [0, \pi), \\
\theta &\in [-\pi, \pi).
\end{align*}
\]
We can rewrite the conditions
\[ x^2 + y^2 + z^2 \leq 1 \text{ and } y \leq x \]
as
\[ (r \leq 1) \text{ and } (r \sin \theta \sin \phi \leq r \cos \theta \sin \phi), \]
which is equivalent (because \( \sin \phi \) is nonnegative for \( \phi \in [0, \pi] \)) to:
\[ (r \leq 1), (\sin \theta \leq \cos \theta) \text{ or, equivalently } r \in [0, 1], \theta \in [-\pi/4, \pi/4]. \]

Therefore, the region \( W \) is defined, in spherical coordinates, by
\[ r \in [0, 1], \quad \phi \in [0, \pi], \quad \theta \in [-\pi/4, \pi/4]. \]

One can also see these limits by drawing the figure containing the ball \( x^2 + y^2 + z^2 \leq 1 \) and the half-space \( y \leq x \) and the corresponding region in the \( xy \) plane consisting of unit disk cut by the half-plane \( y \leq x \).

Now we can compute
\[
\int \int \int_W (1 - x^2 - y^2 - z^2) \, dx \, dy \, dz \\
= \int_0^1 \int_{-\pi/4}^{\pi/4} \int_0^\pi (1 - r^2) r^2 \sin \phi \, dr \, d\theta \, d\phi \\
= \int_0^1 (r^2 - r^4) \, dr \int_0^\pi \sin \phi \, d\phi \int_{-\pi/4}^{\pi/4} \, d\theta = \frac{2}{15} \cdot 2 \cdot \pi = \frac{4\pi}{15}. \]

(b) Find the flux of the vector field \( \mathbf{F} = (x^3 - 3x)i + (y^3 + xy)j + (z^3 - xz)k \) out of the region \( W \).

**Solution.** Let’s first compute
\[
\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} (x^3 - 3x) + \frac{\partial}{\partial y} (y^3 + xy) + \frac{\partial}{\partial z} (z^3 - xz) = 3x^2 + 3y^2 + 3z^2 - 3.
\]

Now, using Gauss’ divergence theorem (and invoking part (a)), we get that the flux of \( \mathbf{F} \) out of the region \( W \) is
\[
\int \int \int_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \int \int \int_W (\nabla \cdot \mathbf{F}) \, dV \\
= \int \int \int_W (3x^2 + 3y^2 + 3z^2 - 3) \, dx \, dy \, dz \\
= (-3) \int \int \int_W (1 - x^2 - y^2 - z^2) \, dx \, dy \, dz \\
= (-3) \frac{4\pi}{15} = -\frac{4\pi}{5}. \]
6. (a) Verify the divergence theorem for the vector field
\[ \mathbf{F} = 4xz \mathbf{i} + y^2 \mathbf{j} + yz \mathbf{k} \]
over the surface of the cube defined by the set of \((x, y, z)\) satisfying \(0 \leq x \leq 1, 0 \leq y \leq 1, \) and \(0 \leq z \leq 1.\)

Solution. First of all, the divergence theorem states that
\[ \iiint_W \text{div} \mathbf{F} \, dV, \]
where \(S\) is the surface of the cube and \(W\) is the cube itself. We evaluate each side and check that they are equal. The right-hand side is, in this case,
\[ \iiint_W \text{div} \mathbf{F} \, dV = \int_0^1 \int_0^1 \int_0^1 (4z + 3y) \, dx \, dy \, dz = 2 + \frac{3}{2} = \frac{7}{2}. \]
The left-hand side can be evaluated by writing
\[ S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6, \]
where
\[
\begin{align*}
S_1 &: \ 0 \leq y, z \leq 1, x = 0, \mathbf{n}_1 = (-1, 0, 0) \\
S_2 &: \ 0 \leq y, z \leq 1, x = 1, \mathbf{n}_2 = (1, 0, 0) \\
S_3 &: \ 0 \leq x, z \leq 1, y = 0, \mathbf{n}_3 = (0, -1, 0) \\
S_4 &: \ 0 \leq s, z \leq 1, y = 1, \mathbf{n}_4 = (0, 1, 0) \\
S_5 &: \ 0 \leq x, y \leq 1, z = 0, \mathbf{n}_5 = (0, 0, -1) \\
S_6 &: \ 0 \leq x, y \leq 1, z = 1, \mathbf{n}_6 = (0, 0, 1).
\end{align*}
\]
Thus, we get
\[
\begin{align*}
\iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \sum_{i=1}^6 \iint_{S_i} \mathbf{F} \cdot \mathbf{n} \, dS \\
&= 0 + \int_0^1 \int_0^1 4z \, dy \, dz + 0 + \int_0^1 \int_0^1 dx \, dz + 0 + \int_0^1 \int_0^1 y \, dx \, dy \\
&= 2 + 1 + \frac{1}{2} = \frac{7}{2}.
\end{align*}
\]
Thus, both sides are equal, so the divergence theorem checks in this case.
(b) Evaluate the surface integral \( \iint_S \mathbf{F} \cdot \mathbf{n} \, dS \), where
\[
\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + z(x^2 + y^2)^2 \mathbf{k},
\]
and \( S \) is the surface of the cylinder \( x^2 + y^2 \leq 1, 0 \leq z \leq 1 \), including the sides and both lids.

**Solution.** Use Gauss’ divergence theorem in space:
\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_W (\text{div} \, \mathbf{F}) \, dx \, dy \, dz.
\]
Here,
\[
\text{div} \, \mathbf{F} = \frac{\partial}{\partial x} (1) + \frac{\partial}{\partial y} (1) + \frac{\partial}{\partial z} z(x^2 + y^2)^2 = (x^2 + y^2)^2.
\]
The region \( W \) is a cylinder, so it is the easiest to evaluate the integral in cylindrical coordinates:
\[
\int_0^1 \int_0^{2\pi} \int_0^1 r \cdot (r^2)^2 \, dr \, d\theta \, dz = \frac{2\pi}{6} = \frac{\pi}{3}.
\]

7. (a) Let \( D \) be the parallelogram in the \( xy \) plane with vertices \((0, 0), (1, 1), (1, 3), (0, 2)\).

Evaluate the integral
\[
\iint_D xy \, dx \, dy.
\]

**Solution.** The region is \( y \)-simple with sides given by the lines \( x = 0 \) and \( x = 1 \), and top and bottom by the lines \( y = x \) and \( y = x + 2 \). Therefore, the integral is
\[
\iint_D xy \, dx \, dy = \int_0^1 \int_x^{x+2} xy \, dy \, dx
= \int_0^1 x \left( \frac{(x+2)^2}{2} - \frac{x^2}{2} \right) \, dx
= \int_0^1 2(x^2 + x) \, dx
= 2 \left[ \frac{3}{3} x^3 + \frac{1}{2} x^2 \right]_0^1 = \frac{5}{3}.
\]
(b) Evaluate
\[
\iiint_D (x^2 + y^2 + z^2)^{1/2} \exp [(x^2 + y^2 + z^2)^2] \, dx \, dy \, dz,
\]
where \( D \) is the region defined by \( 1 \leq x^2 + y^2 + z^2 \leq 4 \) and \( z \geq \sqrt{x^2 + y^2} \).

**Solution.** One should first draw a sketch. The region in question is that between two spheres and inside a cone centered around the \( z \) axis. By drawing a careful figure and using a little trigonometry, one finds that a side of the cone makes an angle of \( \pi/4 \) with the \( z \) axis.

Denoting the integral required by \( I \), we get
\[
I = \int_0^{2\pi} \int_0^{\pi/4} \int_1^2 \rho (e^{\rho^4}) \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta
\]
\[
= 2\pi \left( 1 - \frac{1}{\sqrt{2}} \right) \int_1^2 \exp (\rho^4) \rho^3 \, d\rho
\]
\[
= \frac{\pi}{2} \left( 1 - \frac{1}{\sqrt{2}} \right) \left[ e^{\rho^4} \right]_1^2
\]
\[
= \frac{\pi}{2} \left( 1 - \frac{1}{\sqrt{2}} \right) (e^{16} - e).
\]

8. For each of the questions below, indicate if the statement is true or false. If true, justify (give a brief explanation or quote a relevant theorem from the course), and if false, give an explanation or a counterexample.

(a) If \( P(x, y) = Q(x, y) \), then the vector field \( \mathbf{F} = Pi + Qj \) is a gradient.

**Solution.** This is false. For example,
\[
x i + x j
\]
is not a gradient because it fails to satisfy the cross-derivative test.

(b) The flux of any gradient out of a closed surface is zero.
Solution. This is false. For example, the vector field
\[ F(r) = r \]
is the gradient of \( f(r) = \|r\|^2 / 2 \), yet its flux out of the unit sphere is, either by direct evaluation of the surface integral, or by the divergence theorem, \( 4\pi \).

(c) There is a vector field \( F \) such that \( \nabla \times F = y \hat{j} \).

Solution. This is false. We know that \( \text{div} \, \text{curl} \, F = 0 \) for all vector fields \( F \), but \( \text{div}(y \hat{j}) = 1 \), and so no such \( F \) can exist.

(d) If \( f \) is a smooth function of \((x, y)\), \( C \) is the circle \( x^2 + y^2 = 1 \) and \( D \) is the unit disc \( x^2 + y^2 \leq 1 \), then
\[
\int_C e^{xy} \frac{\partial f}{\partial x} \, dx + e^{xy} \frac{\partial f}{\partial y} \, dy = \iint_D e^{xy} \left[ y \frac{\partial f}{\partial y} - x \frac{\partial f}{\partial x} \right] \, dx \, dy.
\]

Solution. This is true. To see why, let
\[
P(x, y) = e^{xy} \frac{\partial f}{\partial x} \quad \text{and} \quad Q(x, y) = e^{xy} \frac{\partial f}{\partial y},
\]
so that the left-hand side of the expression in the problem is the line integral of \( P \, dx + Q \, dy \). By Green’s theorem we have
\[
\int_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy.
\]

To work out the right-hand side in Green’s theorem, we calculate the partial derivatives:
\[
\frac{\partial Q}{\partial x} = ye^{xy} \frac{\partial f}{\partial y} + e^{xy} \frac{\partial^2 f}{\partial x \partial y} \quad \text{and} \quad \frac{\partial P}{\partial y} = xe^{xy} \frac{\partial f}{\partial x} + e^{xy} \frac{\partial^2 f}{\partial y \partial x}.
\]

By the equality of mixed partials for \( f \), we get the desired equality from Green’s theorem.

(e) For any smooth function \( f(x, y, z) \), we have
\[
\int_0^1 \int_0^x \int_0^{x+y} f(x, y, z) \, dz \, dy \, dx = \int_0^1 \int_0^y \int_0^{x+y} f(x, y, z) \, dz \, dx \, dy.
\]
Solution. The left-hand side represents the volume integral of $f$ over the region $W$ that is between the $xy$ plane and the plane $z = x + y$, and over the region in the $xy$ plane bounded by the $x$ axis, the line $x = 1$, and the line $x = y$. The right-hand side represents the volume integral of $f$ over the region $W$ that is between the $xy$ plane and the plane $z = x + y$, and over the region in the $xy$ plane bounded by the $x$ axis, the line $y = 1$, and the line $x = y$. The student should draw a sketch of these two regions in the $xy$ plane. One sees that these two regions are different, so the two integrals need not be equal. Thus, the assertion is false.

9. Let $W$ be the three-dimensional region defined by

$$x^2 + y^2 \leq 1, \quad z \geq 0, \quad \text{and} \quad x^2 + y^2 + z^2 \leq 4.$$ 

(a) Find the volume of $W$.

Solution 1. The student should draw a sketch. Setting up the volume as a triple integral, one gets

$$\text{Volume} = V = \iiint_W dx \, dy \, dz$$

$$= \iint_D \int_0^{\sqrt{4-x^2-y^2}} dz \, dy \, dx$$

$$= \iint_D \sqrt{4-x^2-y^2} \, dy \, dx,$$

where $D$ is the unit disc in the $xy$ plane. Changing variables to polar coordinates gives

$$V = \int_0^1 \int_0^{2\pi} \sqrt{4-r^2} r \, dr \, d\theta$$

$$= 2\pi \int_0^1 \sqrt{4-r^2} r \, dr.$$

This is now integrated using substitution, and after a little computation, one gets the answer

$$V = \frac{2\pi}{3} (8 - 3\sqrt{3}).$$

Solution 2. The student should draw a sketch. Using evident notation, we have

$$V(W) = V(\text{cap}) + V(\text{cylinder of height } h),$$
where the region is divided into a cylinder with a certain height $h$ and the portion of the sphere above it. Now

$$V(\text{cyl.}) = A(\text{base}) \cdot \text{height} = \pi \cdot h.$$ To determine $h$, note that $h$ is the value of $z$ where the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$ intersect. At such an intersection, $1 + z^2 = 4$, and so $z^2 = 3$, that is, $z = \sqrt{3} = h$. Therefore, 

$$V(\text{cyl.}) = \pi \sqrt{3}.$$ To find the volume of the cap, we think of it as a capped cone minus a cone with a flat top. We can find the volume of the capped cone using spherical coordinates, and we know the volume of the cone with the flat top (the base) is $\frac{1}{3} A(\text{base}) \cdot \text{height}$. First of all, using spherical coordinates,

$$V(\text{capped cone}) = \int_0^{2\pi} \int_0^{\varphi_0} \int_0^2 \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta.$$ To find $\varphi_0$, refer to the figure that we suggested be drawn for this problem, and one sees that for the relevant right triangle,

$$\cos \varphi_0 = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{\sqrt{3}}{2} \quad \text{and so} \quad \varphi_0 = \frac{\pi}{6}.$$ Therefore,

$$V(\text{capped cone}) = \int_0^{2\pi} \int_0^{\pi/6} \int_0^2 \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = \frac{16\pi}{3} \left(1 - \frac{\sqrt{3}}{2}\right).$$ However,

$$V(\text{cone with the flat top}) = \frac{1}{3} \text{Area of base times height}$$

$$= \frac{1}{3} \pi \sqrt{3} = \frac{\pi \sqrt{3}}{3}.$$ Therefore,

$$V(\text{cap}) = \frac{16\pi}{3} - \frac{8\pi \sqrt{3}}{3} - \frac{\pi \sqrt{3}}{3}$$
and finally,

$$V(W) = \frac{16\pi}{3} - \frac{8\pi \sqrt{3}}{3} - \frac{\pi \sqrt{3}}{3} + \frac{3\pi \sqrt{3}}{3}$$

$$= \frac{16\pi}{3} - \frac{6\pi \sqrt{3}}{3} = \frac{2\pi}{3} (8 - 3\sqrt{3}).$$
(b) Find the flux of the vector field $\mathbf{F} = (2x - 3xy)i - yj + 3yzk$ out of the region $W$.

**Solution.** The flux is given by

$$\int\int_{\partial W} \mathbf{F} \cdot d\mathbf{S}.$$ 

To compute this, we use the divergence theorem, which gives

$$\int\int_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \int\int\int_{W} \text{div} \mathbf{F} \, dV = \int\int\int_{W} (2 - 3y - 1 + 3y) \, dV = V(W) = \frac{2\pi}{3} (8 - 3\sqrt{3}).$$

10. Let $f(x, y, z) = xye^{xy}$.

(a) Compute the gradient vector field $\mathbf{F} = \nabla f$.

**Solution.** Using the definition of the gradient, one gets

$$\mathbf{F}(x, y, z) = e^{xy}[(yz + xy^2z)i + (xz + x^2yz)j + xyk].$$

(b) Let $C$ be the curve obtained by intersecting the sphere $x^2 + y^2 + z^2 = 1$ with the plane $x = 1/2$ and let $S$ be the portion of the sphere with $x \geq 1/2$. Draw a figure including possible orientations for $C$ and $S$; state Stokes’ theorem for this region.

**Solution.** The student should draw a figure at this point. One has to choose an orientation for the surface and the curve. For example, if the normal points toward the positive $x$ axis, then the curve should be marked with an arrowhead indicating a counterclockwise orientation when the curve is viewed from the positive $x$-axis. Stokes’ theorem for this region states that for a smooth vector field $\mathbf{K}$:

$$\int_{C} \mathbf{K} \cdot d\mathbf{s} = \int\int_{S} (\nabla \times \mathbf{K}) \cdot d\mathbf{S}.$$

♦
(c) With \( \mathbf{F} \) as in (a) and \( S \) as in (b), let \( \mathbf{G} = \mathbf{F} + (z - y)i + yk \), and evaluate the surface integral
\[
\iint_S (\nabla \times \mathbf{G}) \cdot dS.
\]

**Solution.** We can write \( \mathbf{G} = \mathbf{F} + \mathbf{H}, \) where
\[
\mathbf{H} = (z - y, 0, y).
\]

Thus, we can evaluate the given integral as follows:
\[
\iint_S (\nabla \times \mathbf{G}) \cdot dS = \iint_S (\nabla \times (\mathbf{F} + \mathbf{H})) \cdot dS = \iint_S \nabla \times \mathbf{F} \cdot dS + \iint_S \nabla \times \mathbf{H} \cdot dS.
\]

Because \( \mathbf{F} \) is a gradient, \( \nabla \times \mathbf{F} = 0 \), and so
\[
\iint_S \nabla \times \mathbf{G} \cdot dS = \iint_S \nabla \times \mathbf{H} \cdot dS = \int_C \mathbf{H} \cdot ds
\]
by Stokes' theorem. To evaluate the line integral, we shall parameterize \( C \); we do this by letting
\[
y = \frac{\sqrt{3}}{2} \cos t \quad x = \frac{1}{2} \quad \text{and} \quad z = \frac{\sqrt{3}}{2} \sin t
\]
for \( t \in [0, 2\pi] \). Thus, the line integral of \( \mathbf{H} \) is
\[
\int_C \mathbf{H} \cdot ds = \int_0^{2\pi} \mathbf{H}(c(t)) \cdot c'(t) \, dt
\]
\[
= \frac{3}{4} \int_0^{2\pi} (\sin t - \cos t, 0, \cos t) \cdot (0, -\sin t, \cos t) \, dt
\]
\[
= \frac{3}{4} \int_0^{2\pi} \cos^2 t \, dt.
\]

Remembering that the average of \( \cos^2 \theta \) over the interval from 0 to \( 2\pi \) is \( 1/2 \), we get \( 3\pi/4 \). Thus, the required line integral of \( \nabla \times \mathbf{G} \) is \( 3\pi/4 \).

\[\blacklozenge\]

*The End*