Section 3.4  Fixed points & Functional iteration

Newton's method: \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \)

call this \( F(x_n) \)

So, Newton's method is a functional iteration

\( x_{n+1} = F(x_n) \)

Suppose (in general now) that \( \lim_{n \to \infty} x_n \) exists & \( \lim_{n \to \infty} x_n = a \)

Suppose also that \( F \) is continuous
Then \( \lim_{n \to \infty} F(x_n) = F(\lim_{n \to \infty} x_n) = F(\sigma) \)

\( \lim_{n \to \infty} x_{n+1} = F(x_n) \)

\( \lim_{n \to \infty} x_{n+1} = \sigma \)

So \( \sigma = F(\sigma) \).

Such a point is called a fixed pt. of \( F \).

Very important & interesting problems (e.g. in optimization, algorithm design, diff. equations, ...) can be reduced to finding the fixed pt. of a function \( F \).

Simple (but important case)

\( F : C \to C \) where \( C \) is a closed subset of \( \mathbb{R} \).
Contractive mapping:

\[ |F(x) - F(y)| \leq \lambda |x - y| \]

for some \( \lambda < 1 \)

(can you see why it's called contractive?)

Theorem: Let \( C \subseteq \mathbb{R} \) be closed.
If \( F: C \to C \) is contractive, then \( F \) has a unique fixed pt \( s \). Moreover

\[ s = \lim_{n \to \infty} x_{n+1} \text{ where } x_{n+1} = F(x_n) \]

for any starting pt \( x_0 \in C \).

Proof: want to show that \( (x_n)_{n=0}^{\infty} \) converges.

\[
\begin{align*}
  x_n &= x_0 + (x_1 - x_0) + \cdots + (x_n - x_{n-1}) \\
  &= \left( \sum_{i=1}^{n} (x_i - x_{i-1}) \right) + x_0 \\
  \end{align*}
\]

want this sequence to converge as \( n \to \infty \).
It suffices to show that \( \sum_{i=1}^{n} |x_i - x_{i-1}| \) converges.

But

\[
| x_{i+1} - x_i | = |F(x_i) - F(x_{i-1}) | \\
\text{contraction map} \Rightarrow \leq \lambda | x_i - x_{i-1} | \\
\Rightarrow \leq \lambda^i | x_1 - x_0 | \\
\sum_{i=1}^{\infty} | x_i - x_{i-1} | \leq \sum_{i=1}^{\infty} \lambda^i | x_1 - x_0 | = \frac{1}{1-\lambda} | x_1 - x_0 | \\
\Rightarrow \text{the sequence converges so that } S = \lim_{n \to \infty} x_n \\
\text{and note that } S = F(S).
\]

To see that \( S \) is unique, suppose \( t \) is a different fixed pt. so \( t = F(t) \)

\[
| t - S | = |F(t) - F(S) | \leq \lambda | t - S |
\]

but \( \lambda < 1 \) so \( t = S \).
Exercise: Let \( F(x) = a + b \sin(x) \) for some \( a \in \mathbb{R} \) \& \( b \in \mathbb{R} \). For what values of \( a \) \& \( b \), is \( F \) contractive? In that case, write an iteration to find the fixed point of \( F \).

**Order of Convergence:** Suppose \( \begin{align*} \frac{F(x)}{x} \end{align*} \)

Let \( e_n = x_n - s \)
then \( \lim_{n \to \infty} e_n = 0 \) \& the order of convergence is the smallest integer \( k \geq 1 \) s.t. \( F^{(k)}(s) \neq 0 \).