1 (Some) qual problems and (some) techniques

- (Spring 2008, 1) Let $G$ be a finite group and $H$ a proper subgroup. Show that $G$ is not the set-theoretic union of the conjugates of $H$.
  Consider the intersection and count.

- (Spring 2008, 2) Classify all groups with 99 elements.
  These types of problems are very common, so do a lot of these as practice. Your tools include Sylow, semidirect products, etc.

- (Spring 2008, 3) Let $p$ be prime. If $|G| = p^n$ and $N$ is a normal subgroup, show that $N$ intersects the center of $G$ nontrivially.
  A normal subgroup is a union of conjugacy classes. Count.

- (Spring 2007, 1) Let $p$ be a prime and $G$ a group of order $p^3$.
  (a) Prove that $G$ has a normal subgroup of order $p^2$.
  (b) Assume that $G$ has a cyclic normal subgroup $N$ of order $p^2$ generated by some element $n$. Let $g$ be an element not in $N$.
    i. If the order $|g|$ of $g$ is $p^3$, classify the possible $G$ up to isomorphism.
    ii. If the order $|g|$ of $g$ is $p$, classify the possible $G$ up to isomorphism
  Use Sylow, semidirect products.

- (Fall 2007, 1) Let $G$ be a group of order $240 = 2^4 \cdot 3 \cdot 5$.
  (a) How many $p$-Sylow subgroups might $G$ have, for $p = 2, 3, 5$?
  (b) If $G$ has a subgroup of order 15, show that it has an element of order 15.
  (c) Say $G$ does not have a subgroup of order 15. Show that the number of 3-Sylows is 10 or 40.
  Use Sylow, use Sylow again on the subgroup of order 15, semidirect products.

- (Fall 2006, 2.1) Let $p$ be a prime number. $(\mathbb{Z}/p^2\mathbb{Z})^\times$ denotes the multiplicative group consisting of all congruence classes $\tilde{x} \in \mathbb{Z}/p^2\mathbb{Z}$ such that $\gcd(x, p) = 1$.
  (a) Show that the order of $1 + p$ in $(\mathbb{Z}/p^2\mathbb{Z})^\times$ is equal to $p$. 

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(b) Use (a) to construct a non-abelian group of order $p^3$.

(c) Describe the non-abelian group in (b) via generators and relations.

**Semidirect products, etc.**

- (Fall 2006, 2.2) Let $G$ be a group. Let $r \geq 2$ be an integer. Assume that $G$ contains a non-trivial subgroup $H$ of index $[G : H] = r$. Prove the following.

  (a) If $G$ is simple, then $G$ is finite and $|G|$ divides $r!$.

  (b) If $r \in \{2,3,4\}$, then $G$ cannot be simple.

  (c) For all integers $r \geq 5$, there exist simple groups $G$ which contain non-trivial subgroups $H$ of index $[G : H] = r$. If $G$ is simple, act on cosets of $H$ by multiplication to give an injection $G \to S_n$. This is a common technique when you are dealing with simple groups. Also see Dummit and Foote pp. 201-213.

2 (Some) group things to know

- Basic facts and definitions. (homomorphisms, isomorphism theorems, subgroups, normal subgroups, normalizers, centralizers, quotient groups, cyclic groups, dihedral groups, symmetric groups, etc.)

- $H \leq G$. Given $a, b \in G$, either $aH = bH$ if and only if $a^{-1}b \in H$ or $aH \cap bH = \emptyset$. So cosets partition $G$ and $|aH| = |H|$.


- The kernel of a group homomorphism is a normal subgroup.

- $G$ act on $A$, then for each $g \in G$, we get $\sigma_g : A \to A$. This $\sigma_g$ is a permutation of $A$ and the map $G \to S_n, g \mapsto \sigma_g$ is a homomorphism.

- (Orbit-stabilizer) $|O_x| = [G : G_x] = |G|/|G_x|$.

- Automorphisms

  If $H \leq G$, then $G$ acts by conjugation on $H$ as automorphisms of $H$. Also $G/C_G(H) \cong$ a subgroup of $\text{Aut}(H)$.

  For any $H \leq G$, $N_G(H)/C_G(H) \cong$ a subgroup of $\text{Aut}(H)$.

  $G/Z(G) \cong$ subgroup of $\text{Aut}(G)$.

  $p$ a prime $\iff \text{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}$.

- Isomorphism Theorems

  First Isomorphism Theorem: If $\varphi : G \to H$ is a homomorphism, then $\ker \varphi \leq G$ and $G/\ker \varphi \cong \varphi(G)$. 

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$\varphi$ injective $\iff$ ker $\varphi = 1$

Second Isomorphism Theorem: $A \leq G$, $B \leq G$ and $A \leq N_G(B)$ (or $B \trianglelefteq G$). Then $AB \leq G$ and $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$ and $AB/B \cong A/A \cap B$.

$|AB| = |A||B|/|A \cap B|.$

Third Isomorphism Theorem: $H \trianglelefteq G$ and $K \trianglelefteq G$ with $H \trianglelefteq K$. Then $K/H \trianglelefteq G/H$ and $G/K \cong G/H$.

- **Characteristic subgroups**
  Characteristic subgroups are normal.
  If $H \trianglelefteq G$ is the unique subgroup of a given order, then $H \text{ char } G$.
  $K \text{ char } H$ and $H \trianglelefteq G \implies K \trianglelefteq G$.

- **(Lagrange’s Theorem)** $G$ a finite group, $H \trianglelefteq G$, then $|H| \mid |G|$.

- **(Cauchy’s Theorem)** $G$ a finite group and $p$ a prime such that $p \mid |G|$ then $G$ has an element of order $p$.

- **(Sylow’s Theorem)**
  Sylow $p$-subgroups of $G$ exist.
  If $P \in Syl_p(G)$ and $Q$ any $p$-subgroup of $G$, then $Q \leq gPg^{-1}$.
  $n_p \equiv 1 \pmod{p}$ and $n_p = [G : N_G(P)]$.
  $n_p = 1 \iff P \leq G \iff P \text{ char } G \iff$ All subgroups generated by elements of $p$-power order are $p$-groups.

- **(Fundamental Theorem of Finitely Generated Abelian Groups)**
  $G \cong \mathbb{Z}^{r} \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s}$
  Invariant factors: $n_i \mid n_{i+1}$ for $1 \leq i \leq s - 1$
  Elementary divisors
  If $n$ is the product of distinct primes, the only abelian group of order $n$ is the cyclic group of order $n$, $\mathbb{Z}_n$.
  $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn} \iff (m,n) = 1$.

- **(Class equation)**
  $|G| = |Z(G)| + \sum_i [G : C_G(x_i)]$ (one $x_i$ from each conjugacy class).

- **Commutators**
  $[x,y] = x^{-1}y^{-1}xy$ is called the commutator ($= 1$ iff $x$ and $y$ commute).
  $G' = \langle [x,y] \mid x, y \in G \rangle$ is the commutator subgroup ($= 1$ iff $G$ abelian).
  $xy = yx[x,y]$.
  $H \leq G$ iff $[H,G] \leq H$.
  $G' \text{ char } G$ and $G/G'$ is abelian (the largest abelian quotient).
  If $G' \leq H, H \leq G$, then $G/H$ is abelian.
• Direct products
  If $H, K \trianglelefteq G$ and $H \cap K = 1$, then $HK \cong H \times K$.

• Semidirect products
  Let $K, H$ be groups $\varphi : K \rightarrow \text{Aut}(H)$ a homomorphism. If $\sigma : K \rightarrow K$ is an automorphism of $K$ then
  \[ H \rtimes _{\varphi} K \cong H \rtimes _{\varphi \circ \sigma} K. \]

• $p$-groups
  $|P| = p^a$, $p$ a prime, then:
  \begin{enumerate}
  \item The center of $p$ is non-trivial:
  \item $H \trianglelefteq P$ then $H \cap Z(P) \neq 1$. So every normal subgroup of order $p$ is contained in the center.
  \item $H < P$ then $H < N_P(H)$
  \item Every maximal subgroup of $P$ is of index $p$ and is normal in $P$.
  \end{enumerate}

• Upper central series
  $Z_0(G) = 1$, $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$ (so $Z_{i+1}(G)$ is the preimage in $G$ of the center of $G/Z_i(G)$ under the natural projection).
  $Z_i(G)$ char $G$.

• Nilpotent groups
  $G$ is nilpotent if $Z_n(G) = G$ for some $n$. (So abelian groups are nilpotent).
  If $|P| = p^a$ for prime $a$, then $P$ is nilpotent. ($p$-groups have non-trivial center).
  $|G| = p_1^{a_1} \cdots p_s^{a_s}$, and $P_i \in Syl_{p_i}(G)$. TFAE:
  \begin{enumerate}
  \item $G$ nilpotent;
  \item $H < G$ then $H < N_G(H)$ (normalizers grow);
  \item $P_i \trianglelefteq G$;
  \item $G \cong P_1 \times \ldots \times P_s$.
  \end{enumerate}
  Finite abelian group is direct product of its Sylow subgroups.
  Finite group is nilpotent iff every maximal subgroup is normal
  Subgroups and factor groups of nilpotent groups are nilpotent

• Lower central series
  $G^0 = G$, $G^i = [G, G^{i-1}]$. Then $G^0 \geq G^1 \geq \cdots$
  A group is nilpotent iff $G^n = 1$ for some $n$. 

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• Derived series (Commutator series)
  \[ G^{(0)} = G, \ G^{(i+1)} = [G^{(i)}, G^{(i)}]. \]
  \[ G^{(i)} \text{ char } G. \]
  
  \( G \) is solvable iff \( G^{(n)} = 1 \) for some \( n \).
  
  Nilpotent groups and subgroups of solvable groups are solvable.
  
  If \( G/N \) and \( N \) are solvable, then \( G \) is solvable.