PRACTICE PROBLEMS: FINAL EXAM

20F: WINTER 2015

For the final exam, I would say that the first two things that you should study are the midterm 1 and 2. For the record, difficult problems from each midterm are likely to appear on the final exam. Thus, if there are problems from midterms that you do not know how to do, make sure you know how to do them.

(1) Consider an augmented matrix problem:
\[
\begin{bmatrix}
1 & -1 & 3 & 1 \\
0 & 1 & 4 & 0 \\
0 & 0 & a & b
\end{bmatrix}
\]

Find conditions on \(a, b\) such that the system has infinitely many solutions

(2) Define \(T: \mathbb{R}^3 \rightarrow \mathbb{R}^3\) by
\[
T\left(\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}\right) = \begin{bmatrix}
x_1 + x_2 \\
x_1 + x_3 \\
x_2 - x_3
\end{bmatrix}
\]

(a) Show that \(T\) is a linear transformation
(b) Find a matrix \(A\) such that \(T(x) = Ax\) for all \(x \in \mathbb{R}^3\)
(c) Determine whether \(T\) is one-to-one and justify your answer
(d) Determine whether \(T\) is onto and justify your answer

(3) Suppose \(T\) is a linear transformation from \(\mathbb{R}^4\) to \(\mathbb{R}^3\). Can \(T\) be one-to-one?

(4) Define a function \(T: \mathbb{R}^2 \rightarrow \mathbb{P}_2\) by
\[
T\left(\begin{bmatrix}
a \\
b
\end{bmatrix}\right) = at^2 + bt + 1
\]

Show that \(T\) is not a linear transformation.

(5) Let \(T: \mathbb{R}^3 \rightarrow \mathbb{R}^3\) be a linear transformation such that
\[
T\left(\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}\right) = \begin{bmatrix}1 \\
1 \\
0
\end{bmatrix},
T\left(\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}\right) = \begin{bmatrix}0 \\
1 \\
1
\end{bmatrix},
T\left(\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}\right) = \begin{bmatrix}2 \\
3 \\
2
\end{bmatrix}
\]

Find the standard matrix for \(T^{-1}\)

(6) Let \(B = \left\{ \begin{bmatrix}1 \\
0 \\
1 \\
0 \\
2 \\
4
\end{bmatrix}, \begin{bmatrix}0 \\
1 \\
2 \\
1 \\
4
\end{bmatrix}, \begin{bmatrix}1 \\
1 \\
4
\end{bmatrix} \right\} \). It can be shown that \(B\) is a basis for \(\mathbb{R}^3\).

(a) Find the change-of-coordinates matrix \(P\) from \(B\) to standard basis in \(\mathbb{R}^3\)
   (That is, find \(P\) such that \(P[x]_B = x\) for all \(x \in \mathbb{R}^3\))
(b) Find the change-of-coordinates matrix $P$ from standard basis in $\mathbb{R}^3$ to $\mathcal{B}$ 
(That is, find $Q$ such that $Qx = [x]_\mathcal{B}$ for all $x \in \mathbb{R}^3$)

(7) Let $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$
(a) Find the characteristic polynomial of $A$
(b) Find all eigenvalues of $A$
(c) Find an eigenvector corresponding to each eigenvalue of $A$
(d) Find an invertible matrix $P$, and a diagonal matrix $D$ such that $P^{-1}AP = D$.

(8) We say that an $n \times n$ matrix $A$ is idempotent if $A^2 = A$. Show that if $A$ is idempotent, then all possible eigenvalues of $A$ are 0 and 1.

(9) Consider the following matrix:
$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$
(a) Find the characteristic polynomial of $A$. Conclude that $A$ has two eigenvalues, which are 1 and 2.
(b) Find a basis for the eigenspace of $A$ corresponding to eigenvalue 1
(c) Find a basis for the eigenspace of $A$ corresponding to eigenvalue 2
(d) Explain why $A$ is not diagonalizable

(10) Define a linear transformation $T : \mathbb{P}_3 \to \mathbb{R}^3$ by
$$T(p(t)) = \begin{bmatrix} p(0) \\ p(0) \\ p'(0) + p''(0) \end{bmatrix}$$
(a) Find a basis for $\ker(T)$
(b) Find a basis for $\text{range}(T)$
($p'(0)$ and $p''(0)$ are first and second derivative of $p(t)$ evaluated at 0)

(11) Let $U = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 0 \\ 6 \end{bmatrix} \right\}$. Find an orthonormal basis of $U$.

(12) Let $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$. Find a basis for the orthogonal complement of $W$. 