8.3 The Weierstrass Product Theorem

Now we turn to the fundamental product theorems. The first is the Weierstrass Product Theorem, which is a product analogue of the Mittag-Leffler Theorem. This theorem describes entire functions as a product in terms of the locations of their zeros. The second theorem describes the circumstances in which we can specify the terms product more precisely as we also did with the Mittag-Leffler theorem. This will culminate in the Hadamard product theorem in the next Section. First we give a preliminary theorem, which could have come much earlier.

Theorem 8.15. Suppose that $G(z)$ is an analytic function on a simply connected region $A$ which is nowhere $0$ on $A$. Then there is an analytic function $g(z)$ on $A$ such that

$$G(z) = e^{g(z)}.$$

In particular, if $G(z)$ is an entire function which is nowhere zero, then there is an entire function $g(z)$ such that $G(z) = e^{g(z)}$.

Remark. In essence, this means that $\log(G(z))$ can be defined as a single valued analytic function on $A$; the resulting $\log$ is usually not the principal value, but can be for some $z$ - see exercise below.

Proof. Suppose that $z_0$ is in $A$ and let $G(z_0) = e^c$. For $z$ in $A$, let $\gamma$ be a path from $z_0$ to $z$ in $A$ and define $g(z)$ to be

$$g(z) = c + \int_{\gamma} \frac{G'(\xi)}{G(\xi)} \, d\xi.$$

This definition is path independent since $G'(\xi)/G(\xi)$ is analytic in $A$ by hypothesis and $A$ is simply connected. Thus $g(z)$ is an antiderivative of $G'(z)/G(z)$ and as such is analytic on $A$. We could now give the usual proof that $e^{g(z)} = G(z)$ which proceeds by differentiating $G(z) - 1 \log G(z)$ and finding the derivative is $0$ on $A$, and then checking the value at $z_0$.

However, we have already done all the necessary work in the winding number section and so we will make use of the results here.

Let $\gamma$ be given by the parameterization, $\gamma(t)$ where $a \leq t \leq b$. Define the curve $\Gamma$ by

$$\Gamma(t) = G(\gamma(t)), \quad a \leq t \leq b.$$

By hypothesis, $\Gamma$ does not pass through $0$. In the integral defining $g(z)$, we make the variable change

$$w = G(\xi),$$

see picture.
Although A is simply connected, the image of A under G need not be. Nevertheless, the curve \( \Gamma \) does not pass through the origin.

In terms of the new variable \( w \), we get

\[
g(z) = c + \int \frac{dw}{w}.
\]

Since \( \Gamma \) starts at \( G(z_0) \) and terminates at \( G(z) \), we have seen in Lemma 4.1 that

\[
e^{G(z)} - c = \frac{G(z)}{G(z_0)}.
\]

By the definition of \( c \), this gives \( e^{G(z)} = G(z) \). QED.

**Exercise.** Show that if \( G(z) \) is a non-constant entire function and

\[
G(z) = e^{G(z)}
\]

where \( g(z) \) is an entire function, then there must be values of \( z \) such that

\[
g(z) \neq PV \log \left| G(z) \right|.
\]

**Theorem 8.16 (The Weierstrass Product Theorem).** Suppose we are given a sequence of not necessarily distinct non-zero numbers \( \alpha_1, \alpha_2, \ldots \) tending to \( \infty \). Let \( m \geq 0 \) be an integer. There exists an entire function with a zero of order exactly \( m \) at \( z = 0 \) and with zeros at each of the \( \alpha_j \) with order being the multiplicity of the appearance of \( \alpha_j \) in the sequence \( \{\alpha_j\} \), and with no other zeros.

Further, there are non-negative integers \( N_1, N_2, \ldots \) such that the most general such function \( f(z) \) is of the form,

\[
f(z) = e^{G(z)} \cdot z^m \cdot \prod_{j=1}^{\infty} \left( 1 - \frac{z}{\alpha_j} \right) \cdot \exp \left[ \left( \frac{z}{\alpha_j} \right)^2 + \frac{1}{2} \left( \frac{z}{\alpha_j} \right)^4 + \ldots + \frac{1}{N_j} \left( \frac{z}{\alpha_j} \right)^{N_j} \right],
\]

where the product converges absolutely and uniformly on compact sets \( B \) and \( g(z) \) is an arbitrary entire function.

**Remark.** A given complex number, \( \xi \), may occur more than once in the sequence of \( \alpha_j \), but since the \( \alpha_j \) tend to infinity, only finitely many of
the $\alpha_j$ can equal $\xi$.

**Proof.** We first construct the infinite product. The process takes place in the exponent and the argument is the same as for the Mittag-Leffler Theorem. Let $(S_j)_{j \geq 1}$ be a sequence of positive real numbers such that

$$\sum_{j=1}^{\infty} S_j < \infty.$$ 

We expand the principal value of $\log(1 - w)$ in a power series about $w = 0$,

$$\log(1 - w) = -\sum_{n=1}^{\infty} \frac{1}{n} w^n,$$

the convergence being uniform on $|w| \leq 1/2$. Thus we can find $N_j$ such that

$$|\log(1 - w) + \sum_{n=1}^{N_j} \frac{1}{n} w^n| < S_j \quad \text{for all } w \text{ with } |w| \leq 1/2.$$

Now, given a bounded set $B$ (for instance, when $B$ is compact, $B$ is bounded), we can find $J_0$ so that for $z$ in $B$ and $j \geq J_0$, we have $|z/\alpha_j| \leq 1/2$. Therefore, the series

$$\sum_{j=J_0}^{\infty} \left[ \log(1 - \frac{z}{\alpha_j}) + \sum_{n=1}^{N_j} \frac{1}{n} \left(\frac{z}{\alpha_j}\right)^n \right]$$

converges absolutely and uniformly on $B$ by Inequality (7) and the Weierstrass M-test. In turn, this gives the absolute and uniform convergence of the product,

$$\prod_{j=J_0}^{\infty} \left(1 - \frac{z}{\alpha_j}\right) \cdot \exp \left[ \sum_{n=1}^{N_j} \frac{1}{n} \left(\frac{z}{\alpha_j}\right)^n \right]$$

on $B$. Thus the product

$$f_0(z) = z^m \prod_{j=1}^{\infty} \left(1 - \frac{z}{\alpha_j}\right) \cdot \exp \left[ \sum_{n=1}^{N_j} \frac{1}{n} \left(\frac{z}{\alpha_j}\right)^n \right],$$

is an entire function with all the desired properties.

If $f(z)$ is another such entire function, then

$$G(z) = \frac{f(z)}{f_0(z)}$$

is an entire function with no zeros. By Theorem (8.15), there is an entire function $g(z)$ such that $G(z) = \exp(g(z))$. QED.

**Theorem 8.17.** A **meromorphic function** on $\mathbb{C}$ is the quotient of two
entire functions.

Remark. This is a corollary of the Weierstrass Product Theorem and is one of its most important consequences. Because of this theorem, in several complex variables, a meromorphic function on $\mathbb{C}^n$ is defined to be the quotient of two entire functions, thereby avoiding - at least temporarily - all sorts of unpleasant discussions of the nature of singularities in functions of several variables.

Proof. Suppose that $h(z)$ is a meromorphic function on $\mathbb{C}$ and that $h(z)$ has a pole of order $m$ at $z = 0$ and further poles at $\alpha_1, \alpha_2, \ldots$, where each $\alpha_j$ is repeated according to its multiplicity (order) as a pole. By the Weierstrass Product Theorem, there is an entire function $g(z)$ with these zeros. Then $f(z) = g(z)h(z)$ is entire and we have $h(z) = f(z)/g(z)$.

QED.