Homework 2: Please hand in starred problems by Wed, Oct 23, in class.

Exercise 1:

Let \( \mathcal{L} \) be a non-empty set.

Suppose \( \mathcal{L} \) is a non-empty collection of subsets of \( \mathcal{L} \).

Suppose \( \mathcal{L} \) satisfies

(i) \( \emptyset \in \mathcal{L} \);

(ii) if \( A \in \mathcal{L} \), then \( A^c \in \mathcal{L} \);

(iii) if \( \{ A_n \}_{n=1}^{\infty} \) is a collection of disjoint sets in \( \mathcal{L} \),

then \( \bigcup_{n=1}^{\infty} A_n \) is in \( \mathcal{L} \).

Prove that \( \mathcal{L} \) is a \( \lambda \)-system.

[Comment: in fact the above is an equivalent characterization to the one given in class.]

Exercise 2:

Let \( \mathcal{L} = \{1, 2, 3, 4, 5\} \), \( \mathcal{C} = \{\{1, 2, 3\}, \{2, 3\}\} \).

Find \( \sigma(\mathcal{C}) \).

Exercise 3:

Let \( \mathcal{L} \) be a non-empty set.

A monotone class \( \mathcal{M} \) is a non-empty collection of subsets of \( \mathcal{L} \) that is closed under monotone limits, i.e., \( \mathcal{M} \) is a monotone class if \( \bigcup_{n=1}^{\infty} A_n \in \mathcal{M} \) and \( \bigcap_{n=1}^{\infty} A_n \in \mathcal{M} \), and

\( \bigcup_{n=1}^{\infty} A_n \in \mathcal{M} \) and \( \bigcap_{n=1}^{\infty} A_n \in \mathcal{M} \) for all \( n \).

Show that a \( \sigma \)-algebra is an algebra that is also a monotone class, and conversely, an algebra that is a monotone class is a \( \sigma \)-algebra.

Exercise 4:

Suppose \( \mathcal{P} \) is a \( \Pi \)-system contained in a monotone class \( \mathcal{M} \). Give an example to show that \( \sigma(\mathcal{P}) \) need not be contained in \( \mathcal{M} \). Hint: Consider \( \mathcal{L} \) with a small number of elements.
Exercise 5. Suppose $\mathcal{F}$ is a $\sigma$-algebra of subsets of a non-empty set $\Omega$. Suppose $\mathcal{Q} : \mathcal{F} \to [0,1]$ is a function such that:

(i) $\mathcal{Q}$ is finitely additive on $\mathcal{F}$;
(ii) $0 \leq \mathcal{Q}(A) \leq 1$ for all $A \in \mathcal{F}$ and $\mathcal{Q}(\Omega) = 1$;
(iii) if $\{A_i\}_{i=1}^{\infty}$ are disjoint sets in $\mathcal{F}$ and $\Omega = \bigcup_{i=1}^{\infty} A_i$, then $\sum_{i=1}^{\infty} \mathcal{Q}(A_i) = 1$.

Prove that $\mathcal{Q}$ is a probability measure (i.e., show $\mathcal{Q}$ is countably additive on $\mathcal{F}$).

Exercise 6. Let $\mathcal{S} = \{1, 2, 3, \ldots\}$.
Suppose $\mathcal{A}$ is the algebra of subsets of $\Omega$ containing all sets $A$ such that $A$ or $A^c$ is finite.
Let $\mu : \mathcal{A} \to [0,1]$ be defined by

$$
\mu(A) = \begin{cases} 
0 & \text{if } A \text{ is finite} \\
1 & \text{if } A^c \text{ is finite}.
\end{cases}
$$

(i) Prove that $\mathcal{A}$ is finitely additive on $\mathcal{A}$.

(ii) Give an example of a sequence of sets $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$ such that $A_n \nrightarrow A$ as $n \to \infty$ (i.e., $A_n \nrightarrow A_{n+i}$ for all $n$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$) and $\mu(A_n) \to 0$ as $n \to \infty$. 