Homework #3  Due Wed. Oct. 29 in Class.

1. Let $(\mathcal{S}, \mathcal{F})$ be a measurable space.
   Prove that $A \in \mathcal{F}$ if and only if $1_A : \mathcal{S} \to \mathbb{R}$
   is measurable where $\mathbb{R}$ has the Borel $\sigma$-algebra
   on it.
   Here $1_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$

* 2. Suppose $\Omega = [0, 1]$,
   $\mathcal{F} = \mathcal{B}([0, 1])$,
   $\mathbb{P}$ = Lebesgue
   measure on $([0, 1])$.
   Define
   $X_1(\omega) = 0$ for all $\omega \in \Omega$,
   $X_2(\omega) = 1_{\{\frac{1}{3}\}}(\omega)$ for all $\omega \in \Omega$,
   $X_3(\omega) = 1_A(\omega)$ for all $\omega \in \Omega$
   where $A$ is the set of rational numbers in $[0, 1]$.
   (i) Find $\sigma(X_i)$ for $i = 1, 2, 3$, the $\sigma$-algebras on $\Omega$
       generated by $X_1, X_2, X_3$.
   (ii) Prove that
       $\mathbb{P}(X_1 = X_2 = X_3) = 1$.
       (Hint: You may use the fact that the Lebesgue measure of the
       rationals is zero).
3. Prove that if \( X \) is a random variable, so is \( |X| \).

   Give an example to show that the converse may be false.

4. Exercise 1.3.5 from Durrett.

5. Suppose \( X: \mathbb{R} \to \mathbb{R} \) takes only countably many values in \( \mathbb{R} \). Show that \( X \) is measurable if and only if
   \[
   X^{-1}(\{x\}) \in \mathcal{F} \quad \text{for each } x \in \mathbb{R}.
   \]
   [Here \( \mathbb{R} \) has \( \sigma \)-algebra \( \mathcal{F} \) and \( \mathbb{R} \) has the Borel \( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}) \).]

6. Let \( X \) and \( Y \) be random variables and let \( A \in \mathcal{F} \).

   Prove that
   \[
   Z(w) = \begin{cases} 
   X(w) & \text{if } w \in A, \\
   Y(w) & \text{if } w \notin A
   \end{cases}
   \]
   is a random variable.

   [Here \( X, Y, Z \) are all defined on the measurable space \( (\Omega, \mathcal{F}) \).]