1*. Suppose that \( \{M_t, \mathcal{F}_t, t \geq 0\} \) is a right continuous uniformly integrable martingale and \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, t \geq 0)\) satisfies the usual conditions. Prove that \( M \) is of class \( D \), i.e., \( \{M_T : T \text{ is a real-valued stopping time}\} \) is uniformly integrable.

2*. A standard one-dimensional Brownian motion is a real-valued stochastic process \( \{B_t, t \geq 0\} \) such that

- \( B(0) = 0 \),
- for any positive integer \( n \) and times \( 0 \leq t_1 < t_2 < \ldots < t_n \), \( \{B(t_1), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1})\} \) are independent,
- for each \( 0 \leq s < t < \infty \), \( B(t) - B(s) \) has a normal distribution with mean zero and variance \( t - s \),
- the paths of \( B \) are continuous.

(This is a process with stationary, independent increments).

For each \( t \geq 0 \), let \( \mathcal{H}_t = \sigma\{B(s) : 0 \leq s \leq t\} \), the smallest \( \sigma \)-algebra with respect to which \( B(s) \) is measurable for each \( 0 \leq s \leq t \). Suppose that \((\Omega, \mathcal{F}, P)\) is complete and let \( \mathcal{N} \) denote the \( P \)-null sets in \( \mathcal{F} \). Let \( \mathcal{G}_t = \mathcal{H}_t \vee \mathcal{N} \), the smallest \( \sigma \)-algebra containing \( \mathcal{H}_t \) and \( \mathcal{N} \). Let \( \mathcal{F}_t = \mathcal{G}_t \cap \mathcal{F} \) for all \( t \geq 0 \).

In class we showed that \( \{B(t), \mathcal{H}_t, t \geq 0\} \) and \( \{B_t^2 - t, \mathcal{H}_t, t \geq 0\} \) are martingales.

(a) Prove that \( \{B(t), \mathcal{F}_t, t \geq 0\} \) and \( \{B_t^2 - t, \mathcal{F}_t, t \geq 0\} \) are martingales. Hint: first prove this with \( \mathcal{G}_t \) in place of \( \mathcal{F}_t \).

(b) Let \( T = \inf\{t \geq 0 : B_t > 1\} \). Is \( T \) a stopping time relative to \( \{\mathcal{F}_t, t \geq 0\} \)? Make sure to justify your answer.

(c) Fix \( a < 0 < b \) and let \( S = \inf\{s \geq 0 : B_s \leq a \text{ or } B_s \geq b\} \). Is \( S \) a stopping time relative to \( \{\mathcal{F}_t, t \geq 0\} \)? Use Doob’s stopping theorem to show that \( E[S] < \infty \), \( E[B_S] = 0 \) and to compute the probability that \( B \) hits \( a \) before \( b \). You should also be able to compute \( E[S] \).

(d) (Extra credit) Prove that \( \{M_t = \exp(cB_t - \frac{1}{2}c^2t), \mathcal{F}_t, t \geq 0\} \) is a martingale for any real number \( c \). (You may use results about the moment generating function for the normal distribution.) Prove that \( \lim_{t \to \infty} M_t \) exists a.s. Can you identify this limit?

3*. Let \( B \) be a standard one-dimensional Brownian motion. Let \( X(t) = B(t) - tB(1) \) for all \( 0 \leq t \leq 1 \). The process \( X \) is called a Brownian bridge.
(a) Show that \( \{X(t) : 0 \leq t \leq 1\} \) is a Gaussian process and compute its covariance function \( R(s, t) = \text{Cov}(X(s), X(t)) \) for \( 0 \leq s, t \leq 1 \).

(b) (Extra credit). Show that for \( 0 < t_1 < t_2 < \ldots < t_n < 1 \), the joint distribution of \( (B(t_1), B(t_2), \ldots, B(t_n)) \) given \( |B(1)| \leq \epsilon \), converges to the joint distribution of \( (X(t_1), \ldots, X(t_n)) \), as \( \epsilon \downarrow 0 \).