MATH 280C
HOMEWORK #4, DUE May 6, 2015.

Please hand in the starred problems. In the following, all random variables are defined on the probability space \((\Omega, \mathcal{F}, P)\).

1*. Fractional Brownian motion with a parameter \(H \in (0, 1)\) is a Gaussian process \(\{X(t), t \geq 0\}\) that starts from zero, has expectation zero for all time and has the following covariance function:

\[
R(s, t) = E[X(t)X(s)] = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right),
\]

for all \(s, t \geq 0\). Prove that there is a continuous modification of \(X\). (Extra credit: In fact, there is a modification of \(X\) whose paths are Hölder continuous with exponent \(H - \epsilon\) for all \(\epsilon \in (0, H)\).)

Remark: For \(H > 1/2\), fractional Brownian motion exhibits long range dependence, i.e.,

\[
\sum_{n=1}^{\infty} E[X(1)(X(n + 1) - X(n))] = +\infty.
\]

2*. Suppose that \(B\) is a standard one-dimensional Brownian motion.
(a) Prove that a.s.,

\[
\limsup_{t \to \infty} \frac{B_t}{\sqrt{t}} = +\infty
\]

and

\[
\liminf_{t \to \infty} \frac{B_t}{\sqrt{t}} = -\infty
\]

Hint: Consider positive integer valued \(t\) first and use Kolmogorov’s zero-one law.
(b) Use a transformation of Brownian motion using time inversion to prove that

\[
\limsup_{t \to 0} \frac{B_t}{\sqrt{t}} = +\infty
\]

and

\[
\liminf_{t \to 0} \frac{B_t}{\sqrt{t}} = -\infty.
\]

Use this to argue that a.s. the paths of \(B\) are not Hölder continuous of order one-half at \(t = 0\).