
(b) or (c) $\implies$ (a) is by the completeness of $\mu$.

Consider (a) $\implies$ (b) and (c): Let $R = \bigcup_{j=1}^{\infty} I_j$ with $\{I_j\}$ mutually disjoint $\mu$-finite sets. For each $\mu$-finite set $E_j := E \cap I_j$, by Theorem 1.18 there are an open $U_{mj}$ and a compact $K_{mj}$ such that

$$
\mu(U_{mj} \setminus E_j) \leq \frac{1}{2^m}, \quad \mu(E_j \setminus K_{mj}) \leq \frac{1}{2^m}.
$$

Therefore, $U_m := \bigcup_{j=1}^{\infty} U_{mj}$ is open and $\mu(U_m \setminus E) \leq \frac{1}{m}$, $V := \bigcap_{m=1}^{\infty} U_m \in G_\delta$ and $\mu(V \setminus E) = 0$; $K_j := \bigcap_{m=1}^{\infty} K_{mj}$ is compact and $\mu(K_j \setminus E_j) = 0$, $H := \bigcup_{j=1}^{\infty} K_j \in F_\sigma$ and $\mu(E \setminus H) = 0$.

This is the special case for regularity conditions of Radon measures. Outer regularity by open sets is valid for any measurable set but inner regularity by compact set only works for sets with finite measures.


a. Let $R = Q \cap [0,1)$ and for each $r \in R$ let $E_r = ((E + r) \bigcup (E + r - 1)) \cap [0,1)$ then by invariance under translations of Lebesgue measure, $m(E_r) = m(E)$, and $\{E_r \mid r \in R\}$ consists of pairwise disjoint subsets of $[0,1)$ since $E \subset N$. Hence

$$
\sum_{r \in R} m(E) = \sum_{r \in R} m(E_r) < m([0,1)) = 1.
$$

So $m(E) = 0$.

b. Assume, on the contrary, every $E \cap N_r$ is measurable, from Part a. $\mu(E \cap N_r) = 0$, so

$$
m(E) = \sum_{r \in R} m(E \cap N_r) = 0.
$$

Contradiction!


a. We know $E \subset [0,1)$. Let $E_1 = (E \bigcup (E - 1/2)) \cap [0,1/2)$, where the union is disjoint for $E \subset N$. We have $E_1 \subset [0,1/2)$ and $\mu(E) = \mu(E_1) < 1/2$. Inductively we conclude $\mu(E)$ is bounded by any negative integer power of 2 so $m(E) = 0$.

b. Same as Solution #1.
Solution to Problem 6. Construct a generalized Cantor set $K$ with the sequence $\{\alpha_j\}_{j=1}^\infty$ of numbers in $(0, 1)$ satisfying $\prod_{j=1}^\infty (1 - \alpha_j) = 1 - \alpha$, then let $E = [0, 1] \setminus K$. There are so many possible such sequences in fact if $\sum_{j=1}^\infty \beta_j = -\ln(1 - \alpha)$ is such a convergent series of positive numbers, then let $\alpha_j = 1 - e^{-\beta_j}$. \qed