
Denote the referred measure space by \((X, \mu)\). If \(f_n \to f\) almost uniformly, there is a sequence \(\{E_n\}\) of subspace of \(X\) that \(\mu(E^c_n) < \frac{1}{n}\) and \(f_n \to f\) uniformly on \(E_n\). Then \(f_n \to f\) on \(E := \bigcup_{n=1}^{\infty} E_n\) and \(\mu(E^c) \leq \inf_{n=1}^{\infty} \mu(E^c_n) = 0\). So \(f_n \to f\) on \(X\) \(\mu\)-almost everywhere.

If \(f_n \to f\) almost uniformly, for each \(\delta, \varepsilon > 0\) there is an \(F \subset X\) and an \(N \in \mathbb{N}\) such that \(n \geq N \implies |f_n - f| < \varepsilon\) in \(F\) and \(\mu(F^c) < \delta\). So \(\liminf_{n \to \infty} \mu(\{|f_n - f| \geq \varepsilon\}) \leq \mu(F^c) < \delta\). Let \(\delta \to 0\) we conclude \(f_n \to f\) in measure. \(\square\)


By Theorem 2.26 there is a sequence of continuous function \(\{g_n\}\) on \(X := [a, b]\) that \(\int |f - g| dm < \frac{1}{n}\). By Corollary 2.32 there is a subsequence \(\{g_{n_k}\}\) such that \(g_{n_k} \to f\) a.e. By Egoroff’s theorem, since \(m(x) < \infty\), there exists \(E \subset X\) such that \(m(E^c) < \varepsilon/2\) and \(g_{n_k} \to f\) uniformly on \(E\). So as the uniform limit of continuous functions, \(f|_E\) is continuous. By regularity of Lebesgue measure \(m\), there is a compact set \(K \subset E\) such that \(m(E \setminus K) < \varepsilon/2\). Hence \(f|_K\) is continuous and \(m(K^c) < \varepsilon\). \(\square\)

(a) By Theorem 2.49, integrating in polar coordinates we get
\[
\int |f| dm^2 = \int_0^{\pi/2} \int_0^1 \frac{|\cos(2\theta)|}{r} dr d\theta = \infty.
\]
So \(\int f dm^2\) does not exist. For iterated integrals, we compute
\[
\int_0^1 \int_0^1 f \, dx \, dy = \int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dx \, dy = \int_0^1 \frac{-1}{1 + y^2} \, dy = -\pi/4.
\]
And
\[
\int_0^1 \int_0^1 f \, dy \, dx = \int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy \, dx = -\int_0^1 \int_0^1 \frac{y^2 - x^2}{(x^2 + y^2)^2} \, dy \, dx = \pi/4.
\]
(c) First integrate
\[
\int_0^1 \int_0^1 f \, dy \, dx = \int_0^1 \int_0^{(x-1/2)} \frac{1}{(x - \frac{1}{2})^3} \, dy \, dx = \int_0^1 \frac{|x - \frac{1}{2}|}{(x - \frac{1}{2})^3} \, dx
\]
does not exist. By Tonelli’s theorem,
\[
\int f^+ \, dm^2 = \int_0^1 \int_0^1 f^+ \, dy \, dx = \int_0^1 \int_{1/2}^{x-1/2} \frac{1}{(x - \frac{1}{2})^3} \, dy \, dx = \int_{1/2}^1 \frac{1}{(x - \frac{1}{2})^2} \, dx = +\infty.
\]
And similarly \( \int f^+ \, dm^2 = +\infty \). So \( \int f \, dm^2 \) does not exist. However,
\[
\int_0^1 \int_0^1 f \, dx \, dy = \int_0^1 \int_{(0,1/2-y)}^{(1/2+y,1)} \frac{1}{(x - \frac{1}{2})^3} \, dx \, dy = \int_0^{1/2} 0 \, dy = 0.
\]