
(a) If $dm = f \, d\mu$ then $\forall x \in X = [0, 1]$, for $E = \{x\}$,

$$0 = m(E) = \int_E f \, d\mu = f(x)\mu(E) = f(x).$$

So $f \equiv 0$ on $X$. But then

$$1 = m(X) = \int_X f \, d\mu = \int_X 0 \, d\mu = 0.$$

A contradiction!

(b) Suppose $\mu = \lambda + \rho$, $\lambda \perp m$, $\rho \ll m$. $\lambda \perp m \implies \exists A, B$ that $X = A \sqcup B$, $A$ is $\lambda$-null, $m(B) = 0$. $A \neq \emptyset$, otherwise $m(X) = m(A) + m(B) = 0$. Let $y \in A$, $E = \{y\}$,

$$1 = \mu(\{y\}) = \lambda(\{y\}) + \rho(\{y\}) = 0,$$

since $\rho \ll m$, $\rho(\{y\}) \leq m(\{y\}) = 0$, and $A$ is $\lambda$-null. Contradiction! $\square$


It suffices to assume that $\mu$ is finite and $\nu$ is positive.

Let

$$S = \{ F \in \mathcal{M} \mid F \text{ is } \nu \text{-finite} \}.$$

Let $0 \leq a = \sup_{F \in S} \mu(F) \leq \mu(X) < +\infty$. There is $F_j \in S$ such that $\mu(F_j) \to a$. Let $E = \bigcup_{j=1}^{\infty} F_j \in S$, then $\mu(E) = a$.

$\exists f$ such that $d\mu = f \, d\nu$ on $E$. That is,

$$\forall A \in \mathcal{M}_E, \nu(A) = \int_A f \, d\mu.$$
Extend \( f = +\infty \) on \( E^c \), so \( f \) is defined on \( X \) extended integrable. For any \( B \in \mathcal{M} \), let \( G = B \setminus E \). If \( \mu(G) = 0 \), then \( 0 = \nu(G) = \int_G f \, d\mu \). If \( \mu(G) > 0 \), we must have \( \nu(G) = 0 \) otherwise \( G \cup E \in \mathcal{S} \), \( \mu(G \cup E) > a \) contradicts with the definition of \( a \). Now \( +\infty = \nu(G) = \int_G f \, d\mu \). Hence we have

\[
\nu(B) = \nu(B \cap E) + \nu(G) = \int_{B \cap E} f \, d\mu + \int_G f \, d\mu = \int_B f \, d\mu.
\]

\[\square\]

**Solution to Problem 3 (contributed by Professor B. Li).** Exercise 3.16 in *Real Analysis*, Second Edition by Gerald B. Folland.

We have \( \nu \ll \mu, \nu \ll \lambda, \mu \ll \lambda \), so \( \lambda = \mu + \nu \ll \mu \). Since

\[
1 = \frac{d\mu}{d\lambda} + \frac{d\nu}{d\lambda} = \frac{d\mu}{d\lambda} + f,
\]

where \( \mu \) and \( \lambda \) are mutually absolutely continuous, by Corollary 3.10 \( \frac{d\mu}{d\lambda} > 0 \) a.e. So \( 0 \leq f < 1 \) a.e. Using chain rule twice we get

\[
\frac{d\nu}{d\mu} = \frac{d\nu}{d\lambda} \cdot \frac{d\lambda}{d\mu} = f \left( \frac{d\mu}{d\lambda} \right)^{-1} = \frac{f}{1-f}.
\]

\[\square\]

**Solution to Problem 4.** Exercise 3.31 in *Real Analysis*, Second Edition by Gerald B. Folland.

\( \mu_1 \leq \mu_2 \) since we can take \( E_j = \emptyset \) for \( j > n \).

\( \mu_2 \leq \mu_3 \) since for any countable partition \( E_j \) of \( E \), let \( f = \sum_{j=1}^{\infty} \frac{\nu(E_j)}{\nu(E_j) \lambda E_j} \lambda E_j \), then

\[
\left| \int_E f \, d\nu \right| = \sum_{j=1}^{\infty} \left| \nu(E_j) \right|.
\]

\( \mu_3 \leq \nu \) is evident. Since \( d\nu \ll d|\nu| \), if \( f = \frac{d\nu}{d|\nu|} \) we have \( \left| \int_E f \, d\nu \right| = \left| \nu(E) \right| \), hence \( \mu_3 = \nu \).

To show \( \mu_3 \leq \mu_1 \), for any \( f \) that \( |f| \leq 1 \) and any \( \epsilon > 0 \), we approximate it by simple function \( \phi \) that \( |\phi| \leq 1 \) and \( \int_E |f - \phi| \leq \epsilon \), then

\[
\sum_{j=1}^{n} \left| \nu(E_j) \right| \geq \left| \int_E \phi \, d\nu \right| \geq \left| \int_E f \, d\nu \right| - \epsilon.
\]

Let \( \epsilon \to 0 \) we get \( \mu_3 \leq \mu_1 \).

Henceforth \( \mu_1 = \mu_2 = \mu_3 = |\nu| \).

\[\square\]

$Hf \leq H^*f$ is obvious.

For any $B(r, z) \ni x$, $B(2r, x) \supset B(r, z)$, so
\[
\frac{1}{m(B(r, z))} \int_{B(r, z)} |f(y)| \, dy \leq \frac{2^n}{m(B(2r, x))} \int_{B(2r, x)} |f(y)| \, dy \leq 2^n Hf.
\]

Hence $H^*f \leq 2^n Hf$. \hfill \square


Since $f$ is continuous at $x$, $\forall \varepsilon > 0 \exists \delta > 0$ such that $\forall y \in B(\delta, x)$, $\|f(y) - f(x)\| \leq \varepsilon$, hence if $r \leq \delta$
\[
\frac{1}{m(B(r, x))} \int_{B(r, x)} |f(y) - f(x)| \, dy \leq \varepsilon.
\]

So $x \in L_f$. \hfill \square