5. (a) **Background**: The stock (without dividend paying) return $\mu$ and its volatility $\sigma$ can be computed as follows. Suppose a sequence of historical prices $S_i$ is observed on daily basis. $u_i$ is defined as the continuous compound return $\ln(S_{i+1}/S_i)$ on day $i$ (e.g., $S_{i+1} = S_i e^{u_i}$). Under the assumption that $u_i$ are i.i.d. random variables for all $i$, the daily return $\mu = \mathbb{E}[u_i]$ and the daily volatility $\sigma = \text{Std}[u_i]$ (e.g., standard deviation). The daily risk-free continuous compound return is $r$ (e.g., $\$$1 becomes $\$$e^r$ one day later).

Show that $\text{Var}[\ln(S_T/S_0)] = \sigma^2 T$ (Stock price $S_T$ for day $T$).

(b) **Binomial Tree Construction**: Consider the stock price from the time 0 to $T$ (in days). In the $n$-periods binomial tree, each period corresponds to $\Delta t = T/n$ day. We showed in class that the real probability $p = e^{\frac{\mu \Delta t - d}{u-d}}$, $u = e^{\sigma \sqrt{\Delta t}}$ and $d = e^{-\sigma \sqrt{\Delta t}}$ satisfy (ignore a higher order term $\Delta t^{3/2}$)

$$pS_0u + (1-p)S_0d = S_0e^{\mu \Delta t}$$

and

$$pu^2 + (1-p)d^2 - [pu + (1-p)d]^2 = \sigma^2 \Delta t.$$

In other words, the return and volatility of the binomial model is matched with the real data.

Show that the risk neutral probability is $\tilde{p} = \frac{e^{\mu \Delta t} - d}{u-d}$ and under this risk neutral measure, the volatility of the binomial model does not change, ignoring a higher order term $\Delta t^{3/2}$ (Hint: use the Taylor expansion).

(c) **$S_T$ Distribution under $\tilde{p}$**: Denote $B_i \Delta t$ be the random variable taking 1 when $i$-th coin toss $H$ and $-1$ otherwise. $S_i \Delta t$ is the stock price at the $i$-th period ($S_T$ is at the last period). Given the binomial tree in (a), it is obvious that $\ln(S_T/S_0) = \sigma \sqrt{\Delta t} \sum_{k=1}^n B_i \Delta t$. Question (b) proved that the volatility under $p$ and $\tilde{p}$ are the same, meaning $\text{Var}[\ln(S_T/S_0)] = \sigma^2 T$. Therefore, the central limit theorem (let $n \to \infty$) implies that $\ln(S_T/S_0)$ has the normal distribution $N(a, \sigma^2 T)$ under $\tilde{p}$ for some unknown constant $a$.

Show that $a = \mathbb{E}[\ln(S_T/S_0)] = (r - \frac{\sigma^2}{2})T$. (Hint: use the fact that $\ln(S_T/S_0)$ has Gaussian distribution and the formula $S_0 = e^{-rT} \mathbb{E}[S_T]$, e.g., the discount stock price is martingale under $\tilde{p}$.)

(d) Show that $S_T = S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma^2 Tz}$ with the standard normal random variable $z \sim N(0,1)$ under $\tilde{p}$. $S_T$ is said to satisfy the lognormal distribution under $\tilde{p}$.