Section 1.1-2.3

1. Consider the equation
\[ \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + (\cos^2 t)y = t^3, \]
determine the order of this equation, state whether the equation is ODE or PDE, linear or nonlinear.

**Solution:** It is a second order linear ODE.

2. Verify that
\[ y = e^{t^2} \int_0^t e^{-s^2} ds + e^{t^2} \]
is a solution of the differential equation
\[ y' - 2ty = 1. \]

**Solution:** Since
\[ y = e^{t^2} \int_0^t e^{-s^2} ds + e^{t^2} \]
we differentiate it and get
\[ y' = e^{t^2} \cdot 2t \left( \int_0^t e^{-s^2} ds + e^{t^2} \right) + e^{t^2} \cdot 2t = 2t \left( e^{t^2} \int_0^t e^{-s^2} ds + e^{t^2} \right) + 1 = 2t \cdot y + 1 \]
which is exact the same as
\[ y' - 2ty = 1. \]

3. Briefly Draw a direction field for the differential equation
\[ y' = 3 - 2y. \]

Based on the direction field, determine the behavior of \( y \) as \( t \to \infty \). If this behavior depends on the initial value of \( y \) at \( t = 0 \), describe the dependency.

**Solution:** From the slope field (omit), we can see that \( y \) approaches the equilibrium \( y = 3/2 \) as \( t \to 0 \). If the streamline initially (t=0) start from some point where \( y > 3/2 \), then it will keep decreasing until it approaches \( y = 3/2 \) at \( t = \infty \); if the streamline initially start from \( y < 3/2 \), it keeps growing up to \( y = 3/2 \) at \( t = \infty \).

4. Solve the initial value problem
\[ ty' + 2y = \sin t, \quad y(\pi/2) = 1, \quad t > 0. \]

**Solution:** The original equation is equivalent to
\[ y' + \frac{2}{t} y = \frac{\sin t}{t}. \] (1)
We apply the integrating factor method. First of all, the integrating factor $\mu(t)$ is
\[
\mu(t) = e^{\int \frac{2}{t} \, dt} = t^2.
\]
Second of all, we multiply both sides of (1) by $\mu(t)$, and have:
\[
t^2y' + 2ty = \sin t \cdot t
\]
It turns out that the left hand of (2) is a total derivative and can be written as
\[
(\mu(t)y)' = t \sin t.
\]
Integrating the both sides of (3), we have
\[
\mu(t)y = \int t \sin t \, dt = \sin t - t \cos t + C.
\]
Hence finally the solution is:
\[
y = \frac{\sin t}{t^2} - \frac{\cos t}{t} + \frac{C}{t^2}
\]
and the initial condition $y(\pi/2) = 1$ determines that $C = \pi^2/4 - 1$.

5. Solve the initial value problem
\[
2y' + ty = 2, \quad y(0) = 1.
\]
Show that the solution approaches a limit as $t \to \infty$, and find the limiting value.

**Solution:** Similar as the previous problem, we apply the integrating factor method. The equation can be rewrite as
\[
y' + \frac{t}{2}y = 1.
\]
Then
\[
\mu(t) = e^{\int \frac{t}{2} \, dt} = e^{t^2/4}.
\]
After finding $\mu(t)$,
\[
(\mu(t)y)' = e^{t^2/4} \Rightarrow \mu(t)y = \int_0^t e^{s^2/4} \, ds + C \Rightarrow y = e^{-t^2/4} \int_0^t e^{s^2/4} \, ds + C \cdot e^{-t^2/4}.
\]
Initial condition implies that $C = 1$, so the final solution is:
\[
y = e^{-t^2/4} \int_0^t e^{s^2/4} \, ds + e^{-t^2/4}.
\]
When $t \to +\infty$, $e^{-t^2/4}$ goes to 0. For the first term
\[
\frac{\int_0^t e^{s^2/4} \, ds}{e^{t^2/4}}.
\]
we can apply the L’Hospital’s rule and show that it approaches 0 also.

6. Solve the initial value problem

\[ y' = \frac{2 \cos(2x)}{3 + 2y}, \quad y(0) = -1 \]

and determine where the solution attains its maximum value.

**Solution:** This is a separable ODE which can be separate as

\[(3 + 2y)dy = 2 \cos(2x)dx \Rightarrow 3y + y^2 = \sin(2x) + C \Rightarrow 3y + y^2 = \sin(2x) - 2.\]

To find the max, please follows the steps I did in class.

7-8. Solve the differential equations

\[ \frac{dy}{dx} = \frac{x - e^{-x}}{y + e^y}, \quad \frac{dy}{dx} = \frac{x^2}{1 + y^2}. \]

**Solution:** Both these ODE are separable. For the first one,

\[(y + e^y)dy = (x - e^{-x})dx \Rightarrow \frac{y^2}{2} + e^y = \frac{x^2}{2} + e^{-x} + C; \]

for the second one,

\[(1 + y^2)dy = x^2dx \Rightarrow y + \frac{y^3}{3} = \frac{x^3}{3} + C. \]

9. A tank initially contains 120 L of pure water. A mixture containing a concentration of \( \gamma \) g/L of salt enters the tank at a rate of 2 L/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of \( \gamma \) for the amount of salt in the tank at any time \( t \). Also find the limiting amount of salt in the tank as \( t \to \infty \).

**Solution:** The ODE for this model is:

\[ \frac{dQ}{dt} = 2\gamma - 2 \frac{Q}{120} \Rightarrow Q = 120\gamma - A \cdot e^{-t/60}. \]

since initially \( Q(0) = 0 \), we have

\[ A = 120\gamma \Rightarrow Q = 120\gamma(1 - e^{-t/60}). \]

When \( t \to 0 \), \( Q \) approaches 120\( \gamma \).

10. A young person with no initial capital invests \( k \) dollars per year at an annual rate of return \( r \). Assume that investments are made continuously and that the return is compounded continuously. Determine the sum \( S(t) \) accumulated at any time \( t \).

**Solution:** The equation associated with this compound interest model is

\[ \frac{dS}{dt} = rS + k, \quad S(0) = 0. \]

And the solution is

\[ S = \frac{k}{r}(e^{rt} - 1). \]