4 □ APPLICATIONS OF DIFFERENTIATION

4.1 Maximum and Minimum Values

1. A function $f$ has an **absolute minimum** at $x = c$ if $f(c)$ is the smallest function value on the entire domain of $f$, whereas $f$ has a **local minimum** at $c$ if $f(c)$ is the smallest function value when $x$ is near $c$.

3. Absolute maximum at $s$, absolute minimum at $r$, local maximum at $c$, local minima at $b$ and $r$, neither a maximum nor a minimum at $a$ and $d$.

5. Absolute maximum value is $f(4) = 5$; there is no absolute minimum value; local maximum values are $f(4) = 5$ and $f(6) = 4$; local minimum values are $f(2) = 2$ and $f(1) = f(5) = 3$.

7. Absolute minimum at 2, absolute maximum at 3, local minimum at 4

9. Absolute maximum at 5, absolute minimum at 2, local maximum at 3, local minima at 2 and 4

11. (a) Note: By the Extreme Value Theorem, $f$ must not be continuous; because if it were, it would attain an absolute minimum.
15. \( f(x) = 8 - 3x, x \geq 1 \). Absolute maximum \( f(1) = 5 \); no local maximum. No absolute or local minimum.

17. \( f(x) = x^2, 0 < x < 2 \). No absolute or local maximum or minimum value.

19. \( f(x) = x^2, 0 \leq x < 2 \). Absolute minimum \( f(0) = 0 \); no local minimum. No absolute or local maximum.

21. \( f(x) = x^2, -3 \leq x \leq 2 \). Absolute maximum \( f(-3) = 9 \). No local maximum. Absolute and local minimum \( f(0) = 0 \).

23. \( f(x) = \ln x, 0 < x \leq 2 \). Absolute maximum \( f(2) = \ln 2 \approx 0.69 \); no local maximum. No absolute or local minimum.

25. \( f(x) = 1 - \sqrt{x} \). Absolute maximum \( f(0) = 1 \); no local maximum. No absolute or local minimum.

27. \( f(x) = \begin{cases} 1 - x & \text{if } 0 \leq x < 2 \\ 2x - 4 & \text{if } 2 \leq x \leq 3 \end{cases} \)

Absolute maximum \( f(3) = 2 \); no local maximum. No absolute or local minimum.
29. \( f(x) = 5x^2 + 4x \Rightarrow f'(x) = 10x + 4. \) \( f'(x) = 0 \Rightarrow x = -\frac{2}{5} \), so \(-\frac{2}{5}\) is the only critical number.

31. \( f(x) = x^3 + 3x^2 - 24 \Rightarrow f'(x) = 3x^2 + 6x - 24 = 3(x^2 + 2x - 8). \) \( f'(x) = 0 \Rightarrow 3(x + 4)(x - 2) = 0 \Rightarrow x = -4, 2. \) These are the only critical numbers.

33. \( s(t) = 3t^4 + 4t^3 - 6t^2 \Rightarrow s'(t) = 12t^3 + 12t^2 - 12t. \) \( s'(t) = 0 \Rightarrow 12(t^2 + t - 1) \Rightarrow t = 0 \) or \( t^2 + t - 1 = 0. \) Using the quadratic formula to solve the latter equation gives us
\[
t = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2} \approx 0.618, -1.618. \] The three critical numbers are 0, \(-\frac{1}{2} \sqrt{5}\).

35. \( g(y) = \frac{y - 1}{y^2 - y + 1} \Rightarrow g'(y) = \frac{(y^2 - y + 1)(y - 1) - (y - 1)(2y - 1)}{(y^2 - y + 1)^2} = \frac{y^2 - y + 1 - (2y^2 - 3y + 1)}{(y^2 - y + 1)^2} = \frac{-y^2 + 2y}{(y^2 - y + 1)^2} = \frac{y(y - 2)}{(y^2 - y + 1)^2}. \)
\( g'(y) = 0 \Rightarrow y = 0, 2. \) The expression \( y^2 - y + 1 \) is never equal to 0, so \( g'(y) \) exists for all real numbers.

The critical numbers are 0 and 2.

37. \( h(t) = t^{3/4} - 2t^{1/4} \Rightarrow h'(t) = \frac{3}{4} t^{-1/4} - \frac{2}{4} t^{-3/4} = \frac{3}{4} t^{-3/4}(3t^{1/2} - 2) = \frac{3 \sqrt{t} - 2}{4 \sqrt{t^3}}. \)
\( h'(t) = 0 \Rightarrow 3 \sqrt{t} = 2 \Rightarrow \sqrt{t} = \frac{2}{3} \Rightarrow t = \frac{4}{9}. \) \( h'(t) \) does not exist at \( t = 0, \) so the critical numbers are 0 and \( \frac{4}{9}. \)

39. \( F(x) = x^{4/5}(x - 4)^2 \Rightarrow F'(x) = x^{4/5} \cdot 2(x - 4) + (x - 4)^2 \cdot \frac{4}{5} x^{-1/5} = \frac{1}{5} x^{-1/5}(x - 4)[5x \cdot 2 + (x - 4) \cdot 4] \) \[ = \frac{(x - 4)(14x - 16)}{5x^{1/5}} = \frac{2(x - 4)(7x - 8)}{5x^{1/5}} \]
\( F'(x) = 0 \Rightarrow x = 4, \frac{8}{5}. \) \( F'(0) \) does not exist. Thus, the three critical numbers are 0, \( \frac{8}{5}, \) and 4.

41. \( f(\theta) = 2 \cos \theta + \sin^2 \theta \Rightarrow f'(\theta) = -2 \sin \theta + 2 \sin \theta \cos \theta. \) \( f'(\theta) = 0 \Rightarrow 2 \sin \theta (\cos \theta - 1) = 0 \Rightarrow \sin \theta = 0 \)
or \( \cos \theta = 1 \Rightarrow \theta = n \pi [n \text{ an integer}] \) or \( \theta = 2n \pi. \) The solutions \( \theta = n \pi \) include the solutions \( \theta = 2n \pi, \) so the critical numbers are \( \theta = n \pi. \)

43. \( f(x) = xe^{-3x} \Rightarrow f'(x) = x^2(-3e^{-3x}) + e^{-3x}(2x) = xe^{-3x}(-3x + 2). \) \( f'(x) = 0 \Rightarrow x = 0, \frac{2}{3}. \) \( [e^{-3x} \text{ is never equal to 0}]. \) \( f'(x) \) always exists, so the critical numbers are 0 and \( \frac{2}{3}. \)

45. The graph of \( f'(x) = 5e^{-0.1|x|} \) has 10 zeros and exists everywhere, so \( f \) has 10 critical numbers.
47. \( f(x) = 3x^2 \) \(-12x + 5, [0, 3] \). \( f'(x) = 6x - 12 = 0 \iff x = 2 \). Applying the Closed Interval Method, we find that
\( f(0) = 5, f(2) = -7, \) and \( f(3) = -4 \). So \( f(0) = 5 \) is the absolute maximum value and \( f(2) = -7 \) is the absolute minimum value.

49. \( f(x) = 2x^3 \) \(-3x^2 \) \(-12x + 1, [-2, 3] \). \( f'(x) = 6x^2 \) \(-6x - 12 = 6(x^2 - x - 2) = 6(x - 2)(x + 1) = 0 \iff x = 2, -1 \). \( f(-2) = -3, f(-1) = 8, f(2) = -19, \) and \( f(3) = -8 \). So \( f(-1) = 8 \) is the absolute maximum value and \( f(2) = -19 \) is the absolute minimum value.

51. \( f(x) = x^4 \) \(-2x^2 + 3, [-2, 3] \). \( f'(x) = 4x^3 \) \(-4x = 4x(x^2 - 1) = 4x(x + 1)(x - 1) = 0 \iff x = -1, 0, 1 \).
\( f(-2) = 11, f(-1) = 2, f(0) = 3, f(1) = 2, f(3) = 66 \). So \( f(3) = 66 \) is the absolute maximum value and \( f(\pm 1) = 2 \) is the absolute minimum value.

53. \( f(x) = \frac{x}{x^2 + 1}, [0, 2] \). \( f'(x) = \frac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} = 0 \iff x = \pm 1, \) but \( -1 \) is not in \([0, 2] \). \( f(0) = 0 \), \( f(1) = \frac{1}{2}, f(2) = \frac{2}{5} \). So \( f(1) = \frac{1}{2} \) is the absolute maximum value and \( f(0) = 0 \) is the absolute minimum value.

55. \( f(t) = t \sqrt{4 - t^2}, [-1, 2] \).
\( f'(t) = t \cdot \frac{1}{2}(4 - t^2)^{-1/2} \cdot 2t = \frac{-t^2}{\sqrt{4 - t^2}} + \frac{1 - t^2}{\sqrt{4 - t^2}} = \frac{4 - 2t^2}{\sqrt{4 - t^2}} \)
\( f'(t) = 0 \iff 4 - 2t^2 = 0 \iff t^2 = 2 \iff t = \pm \sqrt{2}, \) but \( t = -\sqrt{2} \) is not in the given interval, \([-1, 2] \).
\( f'(t) \) does not exist if \( 4 - t^2 = 0 \iff t = \pm 2, \) but \( -2 \) is not in the given interval. \( f(-1) = -\sqrt{3}, f(\sqrt{2}) = 2, \) and \( f(2) = 0 \). So \( f(\sqrt{2}) = 2 \) is the absolute maximum value and \( f(-1) = -\sqrt{3} \) is the absolute minimum value.

57. \( f(t) = 2 \cos t + \sin 2t, [0, \pi/2] \).
\( f'(t) = \) \(-2 \sin t + \cos 2t \cdot t = -2 \sin t + 2(1 - 2 \sin^2 t) = -2(2 \sin^2 t + \sin t - 1) = -2(2 \sin t - 1)(\sin t + 1) \).
\( f'(t) = 0 \iff \sin t = \frac{1}{2} \text{ or } \sin t = -1 \iff t = \frac{\pi}{6} \text{ or } t = \frac{5\pi}{6} \). \( f(0) = 2, f(\frac{\pi}{6}) = \sqrt{3} + \frac{1}{2} \sqrt{3} = \frac{3}{2} \sqrt{3} \approx 2.60, \) and \( f(\frac{\pi}{2}) = 0 \).
So \( f(\frac{\pi}{3}) = \frac{3}{2} \sqrt{3} \) is the absolute maximum value and \( f(\frac{\pi}{6}) = 0 \) is the absolute minimum value.

59. \( f(x) = xe^{-x^2/8}, [-1, 4] \). \( f'(x) = x \cdot e^{-x^2/8} \cdot (-\frac{x}{4}) + e^{-x^2/8} \cdot 1 = e^{-x^2/8} \cdot \left(-\frac{x}{4} + 1\right) \).
Since \( e^{-x^2/8} \) is never 0,
\( f'(x) = 0 \iff -x^2/4 + 1 = 0 \iff 1 = x^2/4 \iff x = \pm 2, \) but \( -2 \) is not in the given interval, \([-1, 4] \).
\( f(-1) = e^{-1/8} \approx -0.88, f(2) = 2e^{-1/2} \approx 1.21, \) and \( f(4) = 4e^{-2} \approx 0.54 \). So \( f(2) = 2e^{-1/2} \) is the absolute maximum value and \( f(-1) = -e^{-1/8} \) is the absolute minimum value.

61. \( f(x) = \ln(x^2 + x + 1), [-1, 1] \). \( f'(x) = \frac{1}{x^2 + x + 1} \cdot (2x + 1) = 0 \iff x = -\frac{1}{2} \). Since \( x^2 + x + 1 > 0 \) for all \( x \), the domain of \( f \) and \( f' \) is \( \mathbb{R} \). \( f(-1) = \ln 1 = 0, f(-\frac{1}{2}) = \ln \frac{3}{4} \approx -0.29, \) and \( f(1) = \ln 3 \approx 1.10 \). So \( f(1) = \ln 3 \approx 1.10 \) is the absolute maximum value and \( f(-\frac{1}{2}) = \ln \frac{3}{4} \approx -0.29 \) is the absolute minimum value.
63. \( f(x) = x^n(1 - x)^b \), \( 0 \leq x \leq 1, a > 0, b > 0 \).

\[
f'(x) = x^n \cdot b(1 - x)^{b-1}(-1) + (1 - x)^n \cdot ax^{a-1} = x^{a-1}(1 - x)^{b-1}[x \cdot b(-1) + (1 - x) \cdot a]
\]

\[= x^{a-1}(1 - x)^{b-1}(a - ax - bx)\]

At the endpoints, we have \( f(0) = f(1) = 0 \) [the minimum value of \( f \)]. In the interval \((0, 1)\), \( f'(x) = 0 \iff x = \frac{a}{a + b} \).

\[
f\left(\frac{a}{a + b}\right) = \left(\frac{a}{a + b}\right)^{a} \left(1 - \frac{a}{a + b}\right)^{b} = \frac{a^a}{(a + b)^a} \cdot \frac{b^b}{(a + b)^b} = \frac{a^ab^b}{(a + b)^{a+b}}.
\]

So \( f\left(\frac{a}{a + b}\right) = \frac{a^ab^b}{(a + b)^{a+b}} \) is the absolute maximum value.

65. (a) From the graph, it appears that the absolute maximum value is about \( f(-0.77) = 2.19 \), and the absolute minimum value is about \( f(0.77) = 1.81 \).

(b) \( f(x) = x^5 - x^3 + 2 \implies f'(x) = 5x^4 - 3x^2 = x^2(5x^2 - 3) \). So \( f'(x) = 0 \implies x = 0, \pm \sqrt{\frac{3}{5}} \).

\[
f\left(-\sqrt{\frac{3}{5}}\right) = \left(-\sqrt{\frac{3}{5}}\right)^5 = -\left(\frac{3}{5}\right)^3 + 2 = -\left(\frac{3}{5}\right)^2 \sqrt{\frac{3}{5}} + \frac{3}{5} \sqrt{\frac{3}{5}} + 2 = \frac{9}{25} \sqrt{\frac{3}{5}} + 2 \text{ (maximum)}.
\]

and similarly, \( f\left(\sqrt{\frac{3}{5}}\right) = -\frac{9}{25} \sqrt{\frac{3}{5}} + 2 \text{ (minimum)} \).

67. (a) From the graph, it appears that the absolute maximum value is about \( f(0.75) = 0.32 \), and the absolute minimum value is \( f(0) = f(1) = 0 \) that is, at both endpoints.

(b) \( f(x) = x \sqrt{x - x^2} \implies f'(x) = x \cdot \frac{1 - 2x}{2 \sqrt{x - x^2}} + \sqrt{x - x^2} = \frac{x - 2x^2}{2 \sqrt{x - x^2}} + \frac{3x - 4x^2}{2 \sqrt{x - x^2}} \)

So \( f'(x) = 0 \implies 3x - 4x^2 = 0 \implies x(3 - 4x) = 0 \implies x = 0 \text{ or } \frac{3}{4} \).

\( f(0) = f(1) = 0 \text{ (minimum)} \), and \( f\left(\frac{3}{4}\right) = \frac{3}{4} \sqrt{\frac{3}{4} - \left(\frac{3}{4}\right)^2} = \frac{3}{4} \sqrt{\frac{3}{16}} = \frac{3 \sqrt{3}}{16} \text{ (maximum)} \).

69. The density is defined as \( \rho = \frac{\text{mass}}{\text{volume}} = \frac{1000}{V(T)} \) (in g/cm³). But a critical point of \( \rho \) will also be a critical point of \( V \)

\[
\text{[since } \frac{d\rho}{dT} = -1000V^{-2} \frac{dV}{dT} \text{ and } V \text{ is never 0], and } V \text{ is easier to differentiate than } \rho.
\]

\[
V(T) = 999.87 - 0.06426T + 0.0085043T^2 - 0.00006797^3 \implies V'(T) = -0.06426 + 0.0170086T - 0.0002037T^2.
\]

Setting this equal to 0 and using the quadratic formula to find \( T \), we get

\[
T = \frac{-0.0170086 \pm \sqrt{0.0170086^2 - 4 \cdot 0.0002037 \cdot 0.06426}}{2(-0.0002037)} \approx 3.9665^\circ \text{C or } 79.5318^\circ \text{C}. \text{ Since we are only interested}
\]
in the region 0°C ≤ T ≤ 30°C, we check the density ρ at the endpoints and at 3.9665°C: ρ(0) ≈ \frac{1000}{999.87} ≈ 1.00013;
ρ(30) ≈ \frac{1000}{1003.7628} ≈ 0.99625; ρ(3.9665) ≈ \frac{1000}{999.7447} ≈ 1.000255. So water has its maximum density at about 3.9665°C.

71. Let a = -0.000 032 37, b = 0.000 903 7, c = -0.008 956, d = 0.03629, e = -0.04458, and f = 0.4074.

Then S(t) = at^5 + bt^4 + ct^3 + dt^2 + et + f and S'(t) = 5at^4 + 4bt^3 + 3ct^2 + 2dt + e.

We now apply the Closed Interval Method to the continuous function S on the interval 0 ≤ t ≤ 10. Since S' exists for all t, the only critical numbers of S occur when S'(t) = 0. We use a rootfinder on a CAS (or a graphing device) to find that
S'(t) = 0 when t₁ ≈ 0.855, t₂ ≈ 4.618, t₃ ≈ 7.292, and t₄ ≈ 9.570. The values of S at these critical numbers are
S(t₁) ≈ 0.39, S(t₂) ≈ 0.43645, S(t₃) ≈ 0.427, and S(t₄) ≈ 0.43641. The values of S at the endpoints of the interval are
S(0) ≈ 0.41 and S(10) ≈ 0.435. Comparing the six numbers, we see that sugar was most expensive at t₂ ≈ 4.618
(corresponding roughly to March 1998) and cheapest at t₁ ≈ 0.855 (June 1994).

73. (a) v(r) = k(r₀ - r)r² = kr₀r² - kr³ ⇒ v'(r) = 2kr₀r - 3kr². v'(r) = 0 ⇒ kr(2r₀ - 3r) = 0 ⇒
r = 0 or \frac{2}{3}r₀ (but 0 is not in the interval). Evaluating v at \frac{2}{3}r₀, \frac{4}{3}r₀, and r₀, we get
v(\frac{2}{3}r₀) = \frac{1}{8}kr₀³, v(\frac{4}{3}r₀) = \frac{4}{27}kr₀³,
and v(r₀) = 0. Since \frac{4}{3}r₀ > \frac{3}{2}r₀, v attains its maximum value at r = \frac{2}{3}r₀. This supports the statement in the text.

(b) From part (a), the maximum value of v is \frac{4}{27}kr₀³.

(c) ![Graph of v(r) with maximum at \frac{2}{3}r₀]

75. f(x) = x¹⁰¹ + x⁵¹ + x + 1 ⇒ f'(x) = 101x¹⁰₀ + 51x⁵⁰ + 1 ≥ 1 for all x, so f'(x) = 0 has no solution. Thus, f(x)
has no critical number, so f(x) can have no local maximum or minimum.

77. If f has a local minimum at c, then g(x) = -f(x) has a local maximum at c, so g'(c) = 0 by the case of Fermat’s Theorem
proved in the text. Thus, f'(c) = -g'(c) = 0.

4.2 The Mean Value Theorem

1. f(x) = 5 - 12x + 3x², [1, 3]. Since f is a polynomial, it is continuous and differentiable on ℝ, so it is continuous on [1,3]
and differentiable on (1,3). Also f(1) = -4 = f(3). f'(c) = 0 ⇔ -12 + 6c = 0 ⇔ c = 2, which is in the open
interval (1,3), so c = 2 satisfies the conclusion of Rolle’s Theorem.