1. Consider the following problem: Find two numbers whose sum is 23 and whose product is a maximum.
   (a) Make a table of values, like the following one, so that the sum of the numbers in the first two columns is always 23. On the basis of the evidence in your table, estimate the answer to the problem.

<table>
<thead>
<tr>
<th>First number</th>
<th>Second number</th>
<th>Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>22</td>
<td>22</td>
</tr>
<tr>
<td>2</td>
<td>21</td>
<td>42</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>60</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Use calculus to solve the problem and compare with your answer to part (a).

2. Find two numbers whose difference is 100 and whose product is a minimum.

3. Find two positive numbers whose product is 100 and whose sum is a minimum.

4. Find a positive number such that the sum of the number and its reciprocal is as small as possible.

5. Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

6. Find the dimensions of a rectangle with area 1000 m$^2$ whose perimeter is as small as possible.

7. A model used for the yield $Y$ of an agricultural crop as a function of the nitrogen level $N$ in the soil (measured in appropriate units) is

$$Y = \frac{kN}{1 + N^2}$$

where $k$ is a positive constant. What nitrogen level gives the best yield?

8. The rate (in mg carbon/m$^3$/h) at which photosynthesis takes place for a species of phytoplankton is modeled by the function

$$P = \frac{100I}{I^2 + I + 4}$$

where $I$ is the light intensity (measured in thousands of footcandles). For what light intensity is $P$ a maximum?

9. Consider the following problem: A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?
   (a) Draw several diagrams illustrating the situation, some with shallow, wide pens and some with deep, narrow pens. Find the total areas of these configurations. Does it appear that there is a maximum area? If so, estimate it.
   (b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.
   (c) Write an expression for the total area.

10. Consider the following problem: A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.
   (a) Draw several diagrams illustrating the situation, some short boxes with large bases and some tall boxes with small bases. Find the volumes of several such boxes. Does it appear that there is a maximum volume? If so, estimate it.
   (b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.
   (c) Write an expression for the volume.
   (d) Use the given information to write an equation that relates the variables.
   (e) Use part (d) to write the volume as a function of one variable.
   (f) Finish solving the problem and compare the answer with your estimate in part (a).

11. A farmer wants to fence an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can he do this so as to minimize the cost of the fence?

12. A box with a square base and open top is to have a volume of 32,000 cm$^3$. Find the dimensions of the box that minimize the amount of material used.

13. If 1200 cm$^2$ of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

14. A rectangular storage container with an open top is to have a volume of 10 m$^3$. The length of its base is twice the width. Material for the base costs $10 per square meter. Material for the sides costs $6 per square meter. Find the cost of materials for the cheapest such container.

15. Do Exercise 14 assuming the container has a lid that is made from the same material as the sides.

16. (a) Show that of all the rectangles with a given area, the one with smallest perimeter is a square.
   (b) Show that of all the rectangles with a given perimeter, the one with greatest area is a square.

17. Find the point on the line $y = 4x + 7$ that is closest to the origin.

18. Find the point on the line $6x + y = 9$ that is closest to the point $(-3, 1)$.

19. Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$. 

20. A model used for the yield $Y$ of an agricultural crop as a function of the nitrogen level in the soil (measured in appropriate units) is

$$Y = kN$$

where $k$ is a positive constant. What nitrogen level gives the best yield?