**Exact Equations**

An *exact equation* is a first order differential equation that can be written in the form

\[ M(x,y) + N(x,y)y' = 0, \]

provided that there exists a function \( \psi(x,y) \) such that

\[
\frac{\partial \psi}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial \psi}{\partial y} = N(x,y).
\]

**Note 1:** Often the equation is written in the alternate form of

\[ M(x,y) \, dx + N(x,y) \, dy = 0. \]

**Theorem (Verification of exactness):** An equation of the form

\[ M(x,y) + N(x,y)y' = 0 \]

is an exact equation if and only if

\[
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \]

**Note 2:** If \( M(x) \) is a function of \( x \) only, and \( N(y) \) is a function of \( y \) only, then trivially \( \frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x} \). Therefore, every separable equation,

\[ M(x) + N(y)y' = 0, \]

can always be written, in its standard form, as an exact equation.
The solution of an exact equation

Suppose a function \( \psi(x,y) \) exists such that \( \frac{\partial \psi}{\partial x} = M(x,y) \) and \( \frac{\partial \psi}{\partial y} = N(x,y) \). Let \( y \) be an implicit function of \( x \) as defined by the differential equation

\[
M(x,y) + N(x,y)y' = 0.
\]

(1)

Then, by the Chain Rule of partial differentiation,

\[
\frac{d}{dx} \psi(x, y(x)) = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = M(x,y) + N(x,y)y'.
\]

As a result, equation (1) becomes

\[
\frac{d}{dx} \psi(x, y(x)) = 0.
\]

Therefore, we could, in theory at least, find the (implicit) general solution by integrating both sides, with respect to \( x \), to obtain

\[
\psi(x,y) = C.
\]

Note 3: In practice \( \psi(x,y) \) could only be found after two partial integration steps: Integrate \( M(=\psi_x) \) respect to \( x \), which would recover every term of \( \psi \) that contains at least one \( x \); and also integrate \( N(=\psi_y) \) with respect to \( y \), which would recover every term of \( \psi \) that contains at least one \( y \). Together, we can then recover every non-constant term of \( \psi \).

Note 4: In the context of multi-variable calculus, the solution of an exact equation gives a certain level curve of the function \( z = \psi(x,y) \).
Example: Solve the equation

\[(y^4 - 2) + 4xy^3 y' = 0\]

First identify that \(M(x,y) = y^4 - 2\), and \(N(x,y) = 4xy^3\).

Then make sure that it is indeed an exact equation:

\[\frac{\partial M}{\partial y} = 4y^3 \quad \text{and} \quad \frac{\partial N}{\partial x} = 4y^3\]

Finally find \(\psi(x,y)\) using partial integrations. First, we integrate \(M\) with respect to \(x\). Then integrate \(N\) with respect to \(y\).

\[
\psi(x, y) = \int M(x, y) \, dx = \int (y^4 - 2) \, dx = xy^4 - 2x + C_1(y),
\]

\[
\psi(x, y) = \int N(x, y) \, dy = \int 4xy^3 \, dy = xy^4 + C_2(x).
\]

Combining the result, we see that \(\psi(x,y)\) must have 2 non-constant terms: \(xy^4\) and \(-2x\). That is, the (implicit) general solution is:

\[xy^4 - 2x = C.\]

Now suppose there is the initial condition \(y(-1) = 2\). To find the (implicit) particular solution, all we need to do is to substitute \(x = -1\) and \(y = 2\) into the general solution. We then get \(C = -14\).

Therefore, the particular solution is \(xy^4 - 2x = -14\).
Example: Solve the initial value problem

\[(y \cos(xy) + \frac{y}{x} + 2x)\,dx + (x \cos(xy) + \ln x + e^y)\,dy = 0, \quad y(1) = 0.\]

First, we see that \(M(x, y) = y \cos(xy) + \frac{y}{x} + 2x\) and \(N(x, y) = x \cos(xy) + \ln x + e^y\).

Verifying:

\[
\frac{\partial M}{\partial y} = -xy \sin(xy) + \cos(xy) + \frac{1}{x} = \frac{\partial N}{\partial x} = -xy \sin(xy) + \cos(xy) + \frac{1}{x}
\]

Integrate to find the general solution:

\[
\psi(x, y) = \int \left( y \cos(xy) + \frac{y}{x} + 2x \right)\,dx = \sin(xy) + y \ln x + x^2 + C_1(y),
\]

as well,

\[
\psi(x, y) = \int \left( x \cos(xy) + \ln x + e^y \right)\,dy = \sin(xy) + y \ln x + e^y + C_2(x).
\]

Hence,

\[
\sin xy + y \ln x + e^y + x^2 = C.
\]

Apply the initial condition: \(x = 1\) and \(y = 0\):

\[
C = \sin 0 + 0 \ln (1) + e^0 + 1 = 2
\]

The particular solution is then

\[
\sin xy + y \ln x + e^y + x^2 = 2.
\]
Example: Write an exact equation that has general solution
\[ x^3 e^y + x^4 y^4 - 6y = C. \]

We are given that the solution of the exact differential equation is
\[ \psi(x,y) = x^3 e^y + x^4 y^4 - 6y = C. \]

The required equation will be, then, simply
\[ M(x,y) + N(x,y) y' = 0, \]

such that \[ \frac{\partial \psi}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial \psi}{\partial y} = N(x,y). \]

Since
\[ \frac{\partial \psi}{\partial x} = 3x^2 e^y + 4x^3 y^4, \quad \text{and} \]
\[ \frac{\partial \psi}{\partial y} = x^3 e^y + 4x^4 y^3 - 6. \]

Therefore, the exact equation is:
\[ (3x^2 e^y + 4x^3 y^4) + (x^3 e^y + 4x^4 y^3 - 6) y' = 0. \]
Summary: Exact Equations

\[ M(x,y) + N(x,y) \frac{dy}{dx} = 0 \]

Where there exists a function \( \psi(x,y) \) such that

\[ \frac{\partial \psi}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial \psi}{\partial y} = N(x,y). \]

1. Verification of exactness: it is an exact equation if and only if

\[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \]

2. The general solution is simply

\[ \psi(x,y) = C. \]

Where the function \( \psi(x,y) \) can be found by combining the result of the two integrals (write down each distinct term only once, even if it appears in both integrals):

\[ \psi(x,y) = \int M(x,y) \, dx, \quad \text{and} \]

\[ \psi(x,y) = \int N(x,y) \, dy. \]

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Exercises A-2.2:

1 – 2 Write an exact equation that has the given solution. Then verify that the equation you have found is exact.
1. It has the general solution \( x^2 \tan y + x^3 - y^2 - 3x^4 y^2 = C \).
2. It has a particular solution \( 2xy - \ln xy + 5y = 9 \).

3 – 10 For each equation below, verify its exactness then solve the equation.
3. \( 2x + 2x \cos(x^2) + 2y y' = 0 \)

4. \( 4x^3 y^4 - \frac{2x}{y} - 2x + (4x^4 y^3 + \frac{x^2}{y^2} + 5)y' = 0 \)
5. \( (2x - 2y) + (2y - 2x)y' = 0 \), \( y(10) = -5 \)
6. \( (3x^2 y + y^3 + 4 - ye^{xy}) + (x^3 + 3xy^2 - xe^{xy})y' = 0 \), \( y(2) = 0 \)
7. \( (5 - 2y^2 e^{2x}) + (-5 - 2ye^{2x})y' = 0 \), \( y(0) = -4 \)
8. \( \left( \frac{\sin x}{y^2} + \frac{2x}{y} \right) + \left( \frac{2 \cos x}{y^3} - \frac{x^2}{y^2} \right)y' = 0 \), \( y(0) = 1 \)
9. \( \left( \frac{2xy}{x^4 + 1} + \frac{1}{y^2} \right) + (\arctan(x^2) - \frac{2x}{y^2})y' = 0 \), \( y(1) = 2 \)
10. \( -\sin(x)\sin(2y) + y\cos(x) + (2\cos(x)\cos(2y) + \sin(x))y' = 0 \), \( y(\pi/2) = \pi \)

11 – 13 Find the value(s) of \( \lambda \) such that the equation below is an exact equation. Then solve the equation.
11. \( (2\lambda x^5 y^3 - \frac{1}{x^2}) + (3x^6 y^2 - \lambda)y' = 0 \)
12. \( (\lambda y \sec^2(2xy) - \lambda xy^2) + (2x \sec^2(2xy) - \lambda x^2 y)y' = 0 \)
13. \( (10y^4 - 6xy + 6x^2 \sin(x^3)) + (40xy^3 - 3x^2 + \lambda \cos(x^3))y' = 0 \)
Answers A-2.2:

1. \((2x \tan y + 3x^2 - 12x^3y^2) + (x^2 \sec^2 y - 2y - 6x^4y)\ y' = 0\)

2. \((2y - \frac{1}{x}) + (2x - \frac{1}{y} + 5)\ y' = 0\)

3. \(x^2 + y^2 + \sin(x^2) = C\)

4. \(x^4 y^4 - \frac{x^2}{y} - x^2 + 5y = C\)

5. \(x^2 - 2xy + y^2 = 225\)

6. \(x^3 y + xy^3 + 4x - e^{xy} = 7\)

7. \(5x - 5y - y^2 e^{2x} = 4\)

8. \(-\frac{\cos x}{y^2} + \frac{x^2}{y} = -1\)

9. \(y \arctan(x^2) + \frac{x}{y^2} = \frac{2\pi + 1}{4}\)

10. \(\cos(x)\sin(2y) + y\sin(x) = \pi\)

11. \(\lambda = 3; \quad x^6 y^3 + x^{-1} - 3y = C\)

12. \(\lambda = 2; \quad \tan(2xy) - x^2 y^2 = C\)

13. \(\lambda = 0; \quad 10xy^4 - 3x^2 y - 2\cos(x^3) = C\)