• Turn off and put away your cell phone.
• No electronic devices during the exam.
• No books or other assistance during the exam.
• Show all of your work. No credit will be given for unsupported answers.
• Write your solutions clearly and legibly. No credit will be given for illegible solutions.
• If any question is not clear, ask for clarification.

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Integral Formulas the following integrals might be useful for the test:

\[
\int (ax + b)^n \, dx = \frac{1}{a} \cdot \frac{(ax + b)^{n+1}}{n+1} + C,
\]
\[
\int e^{ax+b} \, dx = \frac{1}{a} \cdot e^{ax+b} + C,
\]
\[
\int \sin(ax + b) \, dx = -\frac{1}{a} \cdot \cos(ax + b) + C,
\]
\[
\int \cos(ax + b) \, dx = \frac{1}{a} \cdot \sin(ax + b) + C,
\]
\[
\int \frac{1}{ax + b} \, dx = \frac{1}{a} \cdot \ln |ax + b| + C,
\]

First Order Ordinary Differential Equations are of the form \( \frac{dy}{dt} = f(y,t) \). In particular,

- if \( \frac{dy}{dt} = g(y)h(t) \), it is called a **separable** equation. To solve it, separate the two variables, and take integrals on both sides:

\[
\frac{1}{g(y)} \, dy = h(t) \, dt \rightarrow \int \frac{1}{g(y)} \, dy = \int h(t) \, dt + C.
\]

- if \( \frac{dy}{dt} + p(t)y = g(t) \), it is called a **linear** equation. To solve it, multiply the two sides by an integrating factor \( \mu(t) \), which can be found by \( \mu = e^{\int p(t) \, dt} \), then the equation becomes:

\[
\mu y' + \mu p y = \mu g \rightarrow (\mu y)' = \mu g \rightarrow \mu y = \int \mu g \, dt + C \rightarrow y = \frac{1}{\mu} \left( \int \mu g \, dt + C \right)
\]

- if \( M(x,y) + N(x,y)y' = 0 \), and \( M_y(x,y) = N_x(x,y) \), it is called an **exact** equation. To solve it, set up a system of partial differential equations:

\[
\phi_x(x,y) = M(x,y), \quad \phi_y(x,y) = N(x,y)
\]

and solve for the **potential** function \( \phi \). Then the solution is \( \phi(x,y) = C \).
Existence and uniqueness of solutions.

- Given an initial value problem
  \[ y' + p(t)y = g(t), \quad y(t_0) = y_0, \]
  if the functions \( p(t) \) and \( g(t) \) are continuous on some interval \((a, b)\), and \( t_0 \in (a, b) \), then this initial value problem has a unique solution.

- Given an initial value problem
  \[ y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0 \]
  if the functions \( p(t), q(t) \) and \( g(t) \) are continuous on some interval \((a, b)\), and \( t_0 \in (a, b) \), then this initial value problem has a unique solution.

Autonomous Equations and stability analysis on their equilibrium solutions, memorize them.

The Second Order Linear, Homogeneous, Constant-coefficient ODEs are of form

\[ ay'' + by' + cy = 0, \]
which can be solved by considering the characteristic equation

\[ ar^2 + br + c = 0. \]

- If the characteristic equation has two distinct real roots \( r_1, r_2 \), then the ODE has two independent solutions \( y_1 = e^{r_1t}, y_2 = e^{r_2t} \);
- If the characteristic equation has two complex roots \( r_{1,2} = \lambda \pm i\mu \), then the ODE has two independent solutions \( y_1 = e^{\lambda t} \cos(\mu t), y_2 = e^{\lambda t} \sin(\mu t) \);
- If the characteristic equation has two equal roots \( r_1 = r_2 = -b/(2a) \), then the ODE has two independent solutions \( y_1 = e^{rt}, y_2 = te^{rt} \);

and the general solution is of the form \( y = c_1y_1 + c_2y_2 \).

Two solutions \( y_1, y_2 \) of the equation \( y'' + p(t)y' + q(t)y = 0 \) are independent if and only if the Wronskian

\[ W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1y'_2 - y'_1y_2 \]

is nonzero.
Reduction of Order: Suppose we know one nonzero solution $y_1(t)$ of the second order linear homogeneous ODE $y'' + p(t)y' + q(t)y = 0$. To find another solution $y_2(t)$ which is independent of $y_1(t)$, we have a few steps

1. Assume $y_2 = v(t)y_1(t)$;
2. Insert $y_2 = v(t)y_1(t)$ into the equation $y'' + p(t)y' + q(t)y = 0$ to have a simplified equation 
   \[ y_1v'' + (2y_1' + py_1)v' = 0; \]
3. Solve the $v$-equation in step 2 by changing it into a system of first order ODEs 
   \[ y_1w' + (2y_1' + py_1)w = 0, \quad v' = w \]
4. After finding the $v$, plug it back into $y_2 = v(t)y_1(t)$, and choose a simple $y_2$ which is independent of $y_1$. Verify the independence of $y_1, y_2$ by calculating the Wronskian.

General Solution of Non-homogeneous Equation Given a non-homogeneous equation $y'' + p(t)y' + q(t)y = g(t)$, and its corresponding homogeneous equation $y'' + p(t)y' + q(t)y = 0$. If $y$ is a homogeneous solution, and $Y$ is a non-homogeneous solution, then $Y \pm y$ are non-homogeneous solutions. According to this fact, the general solution of the non-homogeneous equation can be written in the form of

\[ y = c_1y_1(t) + c_2y_2(t) + Y(t), \]

where $c_1y_1(t) + c_2y_2(t)$ is the general solution of the homogeneous equation, and $Y(t)$ is a particular solution of the non-homogeneous equation.

Method of Undetermined Coefficients Given a non-homogeneous equation $ay'' + by' + cy = g(t)$, the form of its particular solution $Y(t)$ depends on the form of $g(t)$ as follows:

- If $g(t) = P_n(t) = a_nt^n + \cdots + a_0$, then $Y(t) = t^s(A_n t^n + \cdots A_0)$, where $s$ is the number of times that 0 is a root of the characteristic equation;
- If $g(t) = P_n(t)e^{\alpha t}$, then $Y(t) = t^s(A_n t^n + \cdots A_0)e^{\alpha t}$, where $s$ is the number of times that $\alpha$ is a root of the characteristic equation;
- If $g(t) = P_n(t)e^{\alpha t} \sin \beta t$ or $g(t) = P_n(t)e^{\alpha t} \cos \beta t$, then $Y(t) = t^s [ (A_n t^n + \cdots A_0) e^{\alpha t} \cos \beta t + (B_n t^n + \cdots B_0) e^{\alpha t} \sin \beta t ]$, where $s$ is the number of times that $\alpha + i\beta$ is a root of the characteristic equation;
- If $g(t) = P_n(t)e^{\alpha t} \sin \beta t + Q_m(t)e^{\alpha t} \cos \beta t$, then $Y(t) = t^s [ (A_N t^N + \cdots A_0) e^{\alpha t} \cos \beta t + (B_N t^N + \cdots B_0) e^{\alpha t} \sin \beta t ]$, where $N = \max(n, m)$ and $s$ is the number of times that $\alpha + i\beta$ is a root of the characteristic equation.
Method of Variation of Parameters If the functions $p, q, g$ are continuous on an open interval $I$, and if the functions $y_1, y_2$ are a fundamental set of solutions of the homogeneous equation corresponding to the non-homogeneous equation $y'' + p(t)y' + q(t)y = g(t)$, then a particular non-homogeneous solution is

$$Y(t) = -y_1(t) \int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)} \, ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)} \, ds$$

where $t_0$ is any conveniently chosen point in $I$. The general solution is

$$y = c_1 y_1(t) + c_2 y_2(t) + Y(t).$$

Matrix Theory Given a matrix $A$, its eigenvalues $r$ and eigenvectors $\xi$ are defined by the equation $A\xi = r\xi$. To find the eigenvalues $r_1, r_2$, we solve the characteristic equation $|A - I r| = 0$; with the eigenvalues known, we solve the equation $(A - I r)\xi = 0$ for the eigenvectors $\xi^{(1)}, \xi^{(2)}$.

Given a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its inverse is given by $A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, where $|A|$ stands for the determinant of $A$.

Homogeneous System with Constant Coefficients Consider the system $x' = Ax$.

- If the coefficient matrix $A$ has two distinct real eigenvalues $r_1 \neq r_2$, its fundamental solutions are $\{x_1, x_2\} = \{\xi^{(1)}e^{r_1 t}, \xi^{(2)}e^{r_2 t}\}$;
- If the coefficient matrix $A$ has two complex eigenvalues $r_{1,2} = \lambda + i\mu$, and if an eigenvector corresponding to $r_1 = \lambda + i\mu$ is equal to $a + ib$, then its fundamental solutions are $\{u, v\} = \{e^{\lambda t}(a \cos \mu t - b \sin \mu t), e^{\lambda t}(a \sin \mu t + b \cos \mu t)\}$;
- If the coefficient matrix $A$ has a double eigenvalue $r$ with only one linearly independent eigenvector $\xi$, then its fundamental solutions are $\{x_1, x_2\} = \{\xi e^{rt}, \xi e^{rt} + \eta e^{rt}\}$ where $\eta$ is a generalized eigenvector of the matrix $A$ which satisfies $(A - rI)\eta = \xi$.

Diagonalization If a matrix $A$ has two distinct real eigenvalues $r_1, r_2$, and two linearly independent eigenvectors $\xi^{(1)}, \xi^{(2)}$ corresponding to $r_1, r_2$, then the matrix $T = (\xi^{(1)}, \xi^{(2)})$ can diagonalize $A$ as follows

$$T^{-1}AT = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}.$$
**Method of Diagonalization** Consider a non-homogeneous system \( \mathbf{x}' = A\mathbf{x} + \mathbf{g} \). Assume that \( A \) can be diagonalized by some matrix \( T \) as

\[
T^{-1}AT = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}.
\]

Let \( \mathbf{x} = Ty \), then the original system can be converted into a new system \( \mathbf{y}' = (T^{-1}AT)y + T^{-1}\mathbf{g} \) of two uncoupled equations. We can easily solve the uncoupled system \( \mathbf{y}' = (T^{-1}AT)y + T^{-1}\mathbf{g} \) to find the solution \( \mathbf{y} \), then find the solution \( \mathbf{x} \) by \( \mathbf{x} = T\mathbf{y} \).

The **phase portrait** of linear system: Given a linear homogeneous system with constant coefficient \( \mathbf{x}' = A\mathbf{x} \), it has a unique critical point \( \mathbf{x} = 0 \) at the origin. Now let us study the classification and stability of this critical point for different forms of \( A \). To begin with, let us assume \( A \) has two eigenvalues \( r_1, r_2 \) and two linearly independent eigenvectors \( \xi^{(1)}, \xi^{(2)} \) corresponding to \( r_1, r_2 \) respectively. And a particle is traveling by obeying the equation \( \mathbf{x}' = A\mathbf{x} \).

- **If** \( r_1r_2 > 0 \), namely \( r_1, r_2 \) have same signs, then
  - when both negative, \( r_1 < r_2 < 0 \): the particle initially (when \( t \) is very negative) travels along the \( \xi^{(1)} \) direction from 'far far away'. As \( t \to +\infty \), the direction of particle motion tends to along the \( \xi^{(2)} \), and the particle approaches the origin. In this case, the critical point \( \mathbf{x} = 0 \) is called an **asymptotically stable node**.
  - when both positive, \( r_1 > r_2 > 0 \): the particle initially (when \( t \) is very negative) leaves from origin and travels along the \( \xi^{(2)} \) direction. As \( t \to +\infty \), the direction of particle motion tends to along the \( \xi^{(1)} \), and the particle goes to 'far far away'. In this case, the critical point \( \mathbf{x} = 0 \) is called an **unstable node**.

- **If** \( r_1r_2 < 0 \), namely \( r_1, r_2 \) have opposite signs, or \( r_1 < 0 < r_2 \): the particle initially (when \( t \) is very negative) leaves from 'far far away' along the \( \xi^{(1)} \) direction. As \( t \to +\infty \), the direction of particle motion tends to along the \( \xi^{(2)} \), and the particle goes back to 'far far away'. In this case, the critical point \( \mathbf{x} = 0 \) is called an **unstable saddle point**.

- **If** \( r_1, r_2 = \lambda \pm i\mu \), then
  - when \( \lambda < 0 \), the particle travels from 'far far away' to the origin by along a spiral. In this case, the critical point \( \mathbf{x} = 0 \) is called an **asymptotically stable spiral point**.
  - when \( \lambda > 0 \), the particle travels from the origin to 'far far away' by along a spiral. In this case, the critical point \( \mathbf{x} = 0 \) is called an **unstable spiral point**.
  - when \( \lambda = 0 \), the particle travels along an elliptic orbit, and neither approaches or becomes further away from the origin. In this case, the critical point \( \mathbf{x} = 0 \) is called a **stable center**.

- **If** \( r_1 = r_2 = r \), then
  - when \( r \) has TWO linearly independent eigenvector, then each particle trajectory is a straight line from 'far far away' to the origin for \( r < 0 \), and in this case the critical point \( \mathbf{x} = 0 \) is called an **asymptotically stable proper node**; each particle trajectory is a straight line from the origin to 'far far away' for \( r > 0 \), and in this case the critical point \( \mathbf{x} = 0 \) is called an **unstable proper node**.
when \( r < 0 \) has only ONE linearly independent eigenvector \( \xi \), then the particle travels initially from 'far far away' along \( \xi \) direction, and as \( t \to +\infty \) the particle turns 180° in direction and approaches the origin along \( \xi \) direction. In this case, the critical point \( x = 0 \) is called an asymptotically stable improper node. For the case of \( r > 0 \), the trajectory is the same only with opposite direction. In this case, the critical point \( x = 0 \) is called an unstable improper node.

For the section 9.1, the most important point is the correspondence between different cases of the eigenvalues and their phase portraits. You should know where ('where' means the limit of the solution \( x \)) the particle goes when \( t \to -\infty \) and when \( t \to +\infty \), and how ('how' means the direction of the particle motion) the particle goes there.

**Laplace Transform** is defined as

\[
L[f(t)] = F(s) = \int_{0}^{\infty} e^{-st} f(t) dt.
\]

And it has the following properties:

- **Linearity**: \( L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)] = aF(s) + bG(s) \);
- **Differentiation**: \( L[f'(t)] = sL[f(t)] - f(0) = sF(s) - f(0) \);
- **Second differentiation**: \( L[f''(t)] = s^2L[f(t)] - sf(0) - f'(0) = s^2F(s) - sf(0) - f'(0) \);
- **Time shifting**: \( L[f(t - c)H_c(t)] = e^{-cs}L[f] = e^{-cs}F(s) \);
- **Frequency shifting**: \( L[e^{at}f(t)] = F(s - a) \).

where in the Time Shifting formula, \( H_c(t) \) represents the Heaviside function which is defined by

\[
H_c(t) = \begin{cases} 0, & t < c; \\ 1, & t \geq c. \end{cases}
\]

Additionally, there are some elementary Laplace transforms which might be useful in the test.

\[
\begin{align*}
L[1] &= \frac{1}{s}, \quad s > 0 \\
L[e^{at}] &= \frac{1}{s-a}, \quad s > a \\
L[t^n] &= \frac{n!}{s^{n+1}}, \quad n \text{ is a positive integer}, s > 0 \\
L[\sin(at)] &= \frac{a}{s^2 + a^2}, \quad s > 0 \\
L[\cos(at)] &= \frac{s}{s^2 + a^2}, \quad s > 0 \\
L[e^{at}\sin(bt)] &= \frac{b}{(s-a)^2 + b^2}, \quad s > a \\
L[e^{at}\cos(bt)] &= \frac{s-a}{(s-a)^2 + b^2}, \quad s > a \\
L[H_c(t)] &= \frac{e^{-cs}}{s}, \quad s > 0 \\
L[\delta(t-c)] &= e^{-cs}.
\end{align*}
\]
For a piecewise defined function \( f(t) \), it can be written in terms of the Heaviside functions. For example,

\[
\begin{align*}
  f(t) &= \begin{cases} 
  f_1(t), & x \in [0, a) \\
  f_2(t), & x \in [a, \infty) 
  \end{cases} = f_1(t) + [f_2(t) - f_1(t)]H_a(t) \\
  f(t) &= \begin{cases} 
  f_1(t), & x \in [0, a_1) \\
  f_2(t), & x \in [a_1, a_2) \\
  f_3(t), & x \in [a_2, \infty) 
  \end{cases} = f_1(t) + [f_2(t) - f_1(t)]H_{a_1}(t) + [f_3(t) - f_2(t)]H_{a_2}(t)
\end{align*}
\]

To solve an ODE

\[ ay'' + by' + cy = f(t), \quad y(0) = y_0, \quad y'(0) = y'_0, \]

by using Laplace transform, you follow a two-step process:

- Apply Laplace transform on both sides of the equation, and isolate \( Y(s) \) from all other terms;
- Take inverse Laplace transform on \( Y(s) \).

and the diagram here might be helpful.