Def: The Y_i are the coordinate hyperplanes..

In $\mathbb{A}^n$ we also have have coordinate hyperplanes:
if $k[y_1, \dots, y_n]$ is the coordinate ring of $\mathbb{A}^n = k^n$;
then $Z_i := Z(y_i)$ is a coordinate hyperplane.

Def: More generally, given a polynomial of degree 1 in $y_1, \dots, y_n$, its zero locus is called a hyperplane in k^n.

If $L \in S_1$ (i.e., L is homogeneous of degree 1)
$L \neq 0$, then $Z(L)$ is a hyperplane in $\mathbb{P}^n$.

Def: In $\mathbb{A}^n$, a hypersurface of degree d is the zero locus of a polynomial of degree d.

In $\mathbb{P}^n$, a hypersurface of degree d is the zero locus of a homogeneous polynomial of degree d.

Homogenization and dehomogenization:

e.g.: $\mathbb{P}^2$: coordinates (X, Y, Z)

$\mathbb{A}^2$: " (x, y)

$$\varphi_z : \mathbb{A}^2 \hookrightarrow \mathbb{P}^2$$

$$(a, b) \longmapsto (a, b, 1) \qquad \text{image is } U_2 = \{(A, B, C) \mid C \neq 0\}$$

$$\varphi_z^{-1}(A, B, C) = \left(\frac{A}{C}, \frac{B}{C}\right) \longleftarrow (A, B, C) \quad C \neq 0$$

$$\left(x = \frac{X}{Z}, \; y = \frac{Y}{Z}\right)$$

Given a polynomial $P(x, y)$, we can homogenize it to obtain a homogeneous polynomial in X, Y, Z.

e.g.: $xy-1 \rightsquigarrow \frac{X}{Z}\cdot\frac{Y}{Z}-1$ and clear denominators $\rightsquigarrow XY-Z^2$

This was homogenization: denote it by $H: k[x,y] \to k[X,Y,Z]$

Dehomogenization: given a homogeneous polynomial in X, Y, Z, we "evaluate it" at $(x, y, 1)$ to obtain its dehomogenization: $XY-Z^2 \rightsquigarrow xy-1$, denote this by $D: k[X,Y,Z] \longrightarrow k[x,y]$

We have $D \circ H = Id_{k(x,y)}$

However, $H \circ D \neq Id_{k[X,Y,Z]}$, e.g. $XYZ-Z^3$ then $H\circ D(XYZ-Z^3) = XY-Z^2$.

We have $\mathcal{Z}\left(H\left(P(x,y)\right)\right) \cap U_2 = \mathcal{Z}(P(x,y)) \subset U_2$.
(exercise)

<u>Def</u>: A quasi-affine algebraic set is an open subset of an affine algebraic set

<u>Def</u>: A quasi-projective algebraic set is an open subset of a projective algebraic set.

$\boxed{\text{Now assume } k \text{ is algebraically closed.}}$

<u>Nullstellensatz</u>: There is a 1-to-1 inclusion reversing correspondence between algebraic subsets of $\mathbb{A}^n$ and radical ideals in the ring

$A := k[y_1, \dots, y_n]$, given by

$$Y \longmapsto I(Y)$$

$$Z(I) \longleftarrow I$$

Recall: An ideal is radical if $I = \sqrt{I}$, where

$\sqrt{I} := \{ P \in A \mid \exists\, m \in \mathbb{N} \text{ s.t. } P^m \in I \} \supset I$.

Homogeneous nullstellensatz: There is a 1-to-1 inclusion reversing correspondence between projective algebraic sets and homogeneous radical ideals contained in $S_+ := \bigoplus_{d>0} S_d$ given by

$$Y \longmapsto I(Y)$$

$$Z(I) \longleftarrow I$$

where $Z(I)$ is the set of common zeros of all the homogeneous elements of I.

Dimension: $\dim \mathbb{A}^n = \dim k^n = n$

$\dim \mathbb{P}^n = \dim k^{n+1} - 1 = n$

We want to define the dimension of an arbitrary algebraic set (i.e., affine or quasi-affine or projective or quasi-projective),

e.g., if $Y := Z(y_0 + y_1 + y_2 - y_n) \subset k^n$,

then Y should have dimension $n-1$

$Y \supset Y \cap Z(y_0)$ should have dim. $n-2$.

essentially, we could do this n times.

This gives us the idea of defining the dimension of a variety in terms of descending chains of closed subsets.

For this to make sense, we need the idea of irreducibility:

↑intuitively = having one piece

$\supsetneq$
Z_1
$\supsetneq$
Z_2
$\supsetneq$
Z_3

we can get descending chains of closed subsets of any length if we don't require irreducibility $\Rightarrow \dim k \geq n$ $\forall n$.

<u>Def</u>: A topological space Y is called irreducible if $Y \neq \emptyset$ and $\forall$ Y_1, Y_2 non-empty closed subsets of Y, we have

$$Y_1 \cup Y_2 = Y \implies Y_1 = Y \text{ or } Y_2 = Y$$

<u>Definition:</u> A topological space is called noetherian if it satisfies the descending chain condition for closed subsets, i.e., $\forall$ sequence of closed subsets

$$Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_n \supseteq \cdots$$

$$\exists\ n \text{ s.t. } \forall\ r \geq n \quad Y_r = Y_n.$$

<u>Def:</u> The dimension of a topological space Y is

$$\dim Y := \sup\left\{ n \,\middle|\, \begin{array}{l} \exists \text{ chain of distinct irreducible} \\ \text{closed subsets} \\ Y \supseteq Y_0 \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_n \end{array} \right\}$$

Note! You start the indices at 0 because you want a point to have dimension 0 (and k to have dim 1)

Example: $\dim k^n \geq n$: chain

$$k^n = Y_0 \supsetneq \underset{\underset{Z(y_n)}{\shortparallel}}{Y_1} \supsetneq \underset{\underset{Z(y_n, y_{n-1})}{\shortparallel}}{Y_2} \supsetneq \cdots \supsetneq \underset{\underset{Y_n}{\shortparallel}}{\{0\}} = Z(y_1, \dots, y_n)$$

Main goal:

Theorem: (1) If $Y \subset \mathbb{A}^n$ is an irreducible affine variety, then $\dim Y = \dim A(Y)$

$$= \operatorname{trdeg}_k \operatorname{Frac}(A(Y))$$

(2) If $Y \subset \mathbb{P}^n$ is an irreducible projective variety, then $\dim Y = \dim S(Y) - 1$.

Preliminaries for the theorem:

Lemma: $Y \subset \mathbb{A}^n$ or $\mathbb{P}^n$ affine or projective variety.

Then Y irreducible $\iff$ $I(Y)$ is prime.

Proof: Assume $I(Y)$ is prime.

If $Y = Y_1 \cup Y_2$, $Y_1 \neq \phi$, $Y_2 \neq \phi$, Y_1, Y_2 closed.

Then
$$I(Y) = I(Y_1) \cap I(Y_2)$$
$$\cup$$
$$I(Y_1) \cdot I(Y_2)$$

$\Rightarrow$ $I(Y_1) \cdot I(Y_2) \subset I(Y)$

$\Rightarrow$ $I(Y_1) \subseteq I(Y)$ or $I(Y_2) \subseteq I(Y)$

because $I(Y)$ is prime

general fact for affine or projective varieties.

$$Y \subset Z \Rightarrow I(Z) \subset I(Y)$$

So, since $Y_1, Y_2 \subset Y$, we have $I(Y) \subset I(Y_1)$
and $I(Y) \subset I(Y_2)$

So from the previous page:

$$I(Y_1) = I(Y) \text{ or } I(Y_2) = I(Y)$$

$$\Rightarrow Y_1 = Y \text{ or } Y_2 = Y.$$

Now assume Y is irreducible.

If $ab \in I(Y)$, then $Z(ab) \supset Z(I(Y)) = Y$

$$Z(ab) = Z(a) \cup Z(b)$$

$$\Rightarrow (Z(a) \cap Y) \cup (Z(b) \cap Y) = Y$$

Y irreducible $\Rightarrow$ $Z(a) \cap Y = Y$ or $Z(b) \cap Y = Y$

$\Rightarrow$ $Z(a) \supset Y$ or $Z(b) \supset Y$

$\Rightarrow$ $a \in I(Y)$ or $b \in I(Y)$. $\square$

<u>Dimension of a commutative ring</u> (all rings have identity elements, denoted 1)

<u>Def</u>: R a comm. ring. $\mathfrak{p} \subset R$ prime ideal

(1) The height of $\mathfrak{p}$ is

$$\text{height}(\mathfrak{p}) := \sup\{n \mid \exists\ \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subseteq \mathfrak{p}\}$$

$\mathfrak{p}_i$ all prime

(2) The Krull dimension of R is

$$\dim(R) := \sup\{\text{height}(\mathfrak{p}) \mid \mathfrak{p} \subset R \text{ prime}\}$$