More concretely: $\mathcal{F}_x := \coprod_{x \in U \subset X} \mathcal{F}(U) \Big/ \sim$

$$= \left\{ (U, s) \mid x \in U \subset X,\ s \in \mathcal{F}(U) \right\} \Big/ \sim$$

where $(U, s) \sim (V, t) \iff \exists_{x \in} W \subset U \cap V$
s.t. $s|_W = t|_W$

Terminology: For $U \subset X$ and $s \in \mathcal{F}(U)$ and $x \in U$, we call the image of (U, s) in $\mathcal{F}_x$ the germ of s at x and we denote it $s(x)$.

Def: The sheaf $\tilde{\mathcal{F}}$ associated to a presheaf $\mathcal{F}$ is the unique (up to isomorphism) sheaf with a morphism

$f : \mathcal{F} \longrightarrow \tilde{\mathcal{F}}$ s.t., for any sheaf $\mathcal{G}$, any morphism of presheaves $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ factors uniquely through $\tilde{\mathcal{F}}$:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{G} \\ \tilde{\mathcal{F}} \arrow[ur, dashed, "\exists! \tilde{\varphi}"'] & \end{tikzcd}$$

$\exists! \tilde{\varphi}$ which makes the diagram commute.

<u>Def</u>: A morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is the data of maps $\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ for all open sets U s.t. $\forall\ V \subset U$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\text{restriction}"'] & \mathcal{G}(U) \arrow[d, "\text{restriction}"] \\ \mathcal{F}(V) \arrow[r, "\varphi(V)"] & \mathcal{G}(V) \end{tikzcd}$$

A morphism of sheaves is a morphism of the underlying presheaves.

We construct the sheaf associated to a presheaf using the "espace étalé" of a presheaf.

<u>Def</u>: The espace étalé of a presheaf $\mathcal{F}$ on X is the set $\overline{\mathcal{F}} := \coprod_{x \in X} \mathcal{F}_x$ with a topology defined below.

$$\begin{array}{ccc} \overline{\mathcal{F}} & := \coprod_{x\in X} & \mathcal{F}_x \\ \text{projection } \pi \downarrow & & \downarrow \\ X & \ni & x \end{array}$$

Given $U \subseteq X$ any $s \in \mathcal{F}(U)$ defines a section of π:

$$s : U \longrightarrow \coprod_{x \in U} \mathcal{F}_x \subset \overline{\mathcal{F}} \qquad x \longmapsto s(x) \in \mathcal{F}_x$$

We endow $\overline{\mathcal{F}}$ with the topology whose open sets are unions of sets of the form $s(U) \subset \overline{\mathcal{F}}$

<u>Def</u>: The sheaf $\tilde{\mathcal{F}}$ can be defined as the sheaf of continuous sections of π for the above topology.

<u>More concretely</u>: $U \subset X$ open, we describe $\tilde{\mathcal{F}}(U)$.

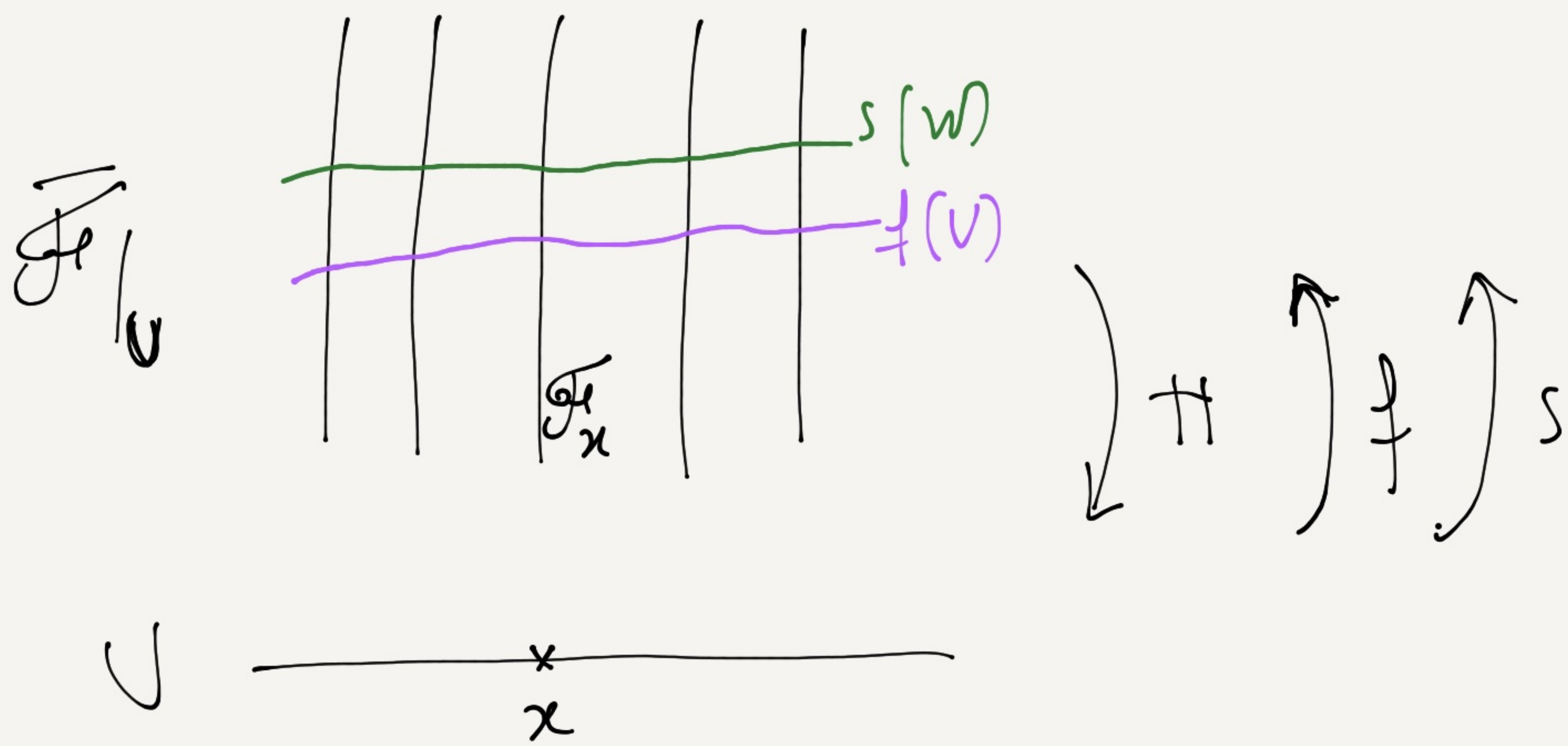
$$\tilde{\mathcal{F}}(U) := \left\{ f : U \longrightarrow \overline{\mathcal{F}} \;\middle|\; f \text{ continuous and } \pi \circ f = Id_U \right\}$$
$$= \left\{ f : U \longrightarrow \overline{\mathcal{F}} \;\middle|\; f \text{ continuous and } \forall x \in U, f(x) \in \mathcal{F}_x \right\}$$

Understanding the continuity: fix $U \subset X$

$f : U \longrightarrow \overline{\mathcal{F}}$ continuous if $\forall\, V \subset \overline{\mathcal{F}}$ open, $f^{-1}(V) \subset U$ is open.

WLOG we can assume $V = s(W)$ for some $W \subset U$ open and $s \in \mathcal{F}(W)$.

What does it mean to say $f^{-1}(s(W))$ is open in U?



$$f^{-1}(s(W)) = \{x \in U \mid f(x) \in s(W)\}$$
$$= \{x \in U \mid f(x) = s(x)\}$$
$$= s^{-1}(f(W))$$

f is continuous $\iff$ $\forall\ x \in W \subset U$, $\forall\, s \in \mathcal{F}(W)$

$$f^{-1}(s(W)) = \{x \in W \,/\, f(x) = s(x)\} \text{ is open.}$$

$\iff$ $\forall\ (W, s)$ as above $f^{-1}(s(W))$ is an open neighborhood of all its points.

$\Updownarrow$

$\forall\ y \in f^{-1}(s(W))$, $\exists$ open $y \in V_y \subset f^{-1}(s(W))$

$\Updownarrow$

$\forall z \in V_y\ f(z) = s(z)$

$\Updownarrow$

$\forall\ y \in f^{-1}(s(W))$, $\exists$ open V_y s.t. $f|_{V_y} = s|_{V_y}$

So f is continuous iff U can be covered with open sets U' s.t. $\forall\ U'$ $\exists\ s \in \mathcal{F}(U')$ with $f|_{U'} = s|_{U'}$.

<u>Definition</u>: The morphism of presheaves

$$\mathcal{F} \longrightarrow \tilde{\mathcal{F}}$$

is defined by sending $s \in \mathcal{F}(U)$ to the section of $\tilde{\mathcal{F}}$ defined by s.

<u>Verification of the universal property</u>:

Let $\mathcal{G}$ be a sheaf with a morphism of presheaves

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \\
\tilde{\mathcal{F}} \arrow[ur, dashed, "\tilde{\varphi}"'] &
\end{tikzcd}$$

find $\tilde{\varphi}$ and show it is unique

Uniqueness of $\tilde{\varphi}$:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \\ \tilde{\mathcal{F}} \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

Given $U \subset X$ and $f \in \tilde{\mathcal{F}}(U)$

$\exists$ open covering $U = \bigcup_{i \in I} W_i$ and sections $s_i \in \mathcal{F}(W_i)$

s.t. $f|_{W_i} = s_i \quad \forall i$, i.e., $\forall x \in W_i$, $f(x) = s_i(x)$.

Then $\forall i$ we have $\varphi(s_i) = \tilde{\varphi}(f|_{W_i}) = \tilde{\varphi}(f)|_{W_i}$

So $\tilde{\varphi}(f)|_{W_i}$ is determined by $\varphi(s_i)$

Since $\mathcal{G}$ is a sheaf $\tilde{\varphi}(f)$ is determined

by $\{\tilde{\varphi}(f)|_{W_i} \mid i \in I\} = \{\varphi(s_i) \mid i \in I\}$

$\Rightarrow$ uniqueness.

Existence of $\tilde{\varphi}$: $\quad \tilde{\varphi} : \tilde{\mathcal{F}} \longrightarrow \mathcal{G}$

open $U \subset X$ $\quad f \in \tilde{\mathcal{F}}(U)$, we define $\tilde{\varphi}(f) \in \mathcal{G}(U)$.

$\exists$ open covering $U = \bigcup_{i \in I} W_i$ and $s_i \in \mathcal{F}(W_i)$ s.t.

$\forall i \quad f|_{W_i} = s_i$

So $\tilde{\varphi}(f|_{W_i}) = \varphi(s_i) \Rightarrow \tilde{\varphi}(f)|_{W_i} = \varphi(s_i) \in \mathcal{G}(W_i)$

If we know $\boxed{\varphi(s_i)|_{W_i \cap W_j} = \varphi(s_j)|_{W_i \cap W_j}}$

$\forall i, j$, then because $\mathcal{G}$ is a sheaf, $\exists!$ section $\tilde{\varphi}(f) \in \mathcal{G}(U)$ s.t. $\tilde{\varphi}(f)|_{W_i} = \varphi(s_i)$