$$\varphi^{-1}(U_f) = \left\{ x \in X \;\middle|\; \begin{array}{l} \exists\ V_x \subset X \text{ open neighborhood of } x \\ \exists\ h \in \mathcal{O}_X(V_x) \text{ s.t. } h \cdot \alpha(f)|_{V_x} = 1 \end{array} \right\}$$

$\forall\ x \in \varphi^{-1}(U_f)$, $\varphi^{-1}(U_f) \supset V_x$ because

$$\forall\, y \in V_x \qquad h(y) \cdot (\alpha(f))(y) = 1$$

$\Rightarrow\ \varphi^{-1}(U_f)$ is open.

Next we define the morphism of sheaves

$$\varphi^{\#} : \mathcal{O}_A \longrightarrow \varphi_* \mathcal{O}_X \quad \text{on } \operatorname{Spec} A$$

we already have

$$\begin{array}{rl} \varphi^{\#}(\operatorname{Spec} A) : A & \longrightarrow (\varphi_* \mathcal{O}_X)(\operatorname{Spec} A) \\ & \xrightarrow{\alpha} \quad \overset{\shortparallel}{\mathcal{O}_X}(X) \end{array}$$

We first define $\varphi^{\#}$ on basic open sets:

choose $f \in A$ $\qquad \mathcal{O}_A(U_f) = A[f^{-1}]$

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d] \arrow[dr, "\alpha|_{U_f}"] & \mathcal{O}_X(X) = (\varphi_* \mathcal{O}_X)(\operatorname{Spec} A) \arrow[d] \\ A[f^{-1}] \arrow[r, dotted, "?"] & \mathcal{O}_X(\varphi^{-1}(U_f)) = (\varphi_* \mathcal{O}_X)(U_f) \end{tikzcd}$$

<u>recall</u>: $\varphi^{-1}(U_f) = \{x \in X \mid \alpha(f)(x) \in \mathcal{O}_{X,x} \text{ is invertible}\}$

$$\| \\ X_{\alpha(f)}$$

We need to show that $\alpha|_{U_f}$ factors through $A[f^{-1}]$.

By the universal property of localizations, we need to show that $\alpha(f)|_{\varphi^{-1}(U_f) = X_{\alpha(f)}}$ is invertible in $\mathcal{O}_X(\varphi^{-1}(U_f))$.

$$\mathcal{O}_X(\varphi^{-1}(U_f)) = \mathcal{O}_X(X_{\alpha(f)})$$

We saw that $\forall\ x \in X_{\alpha(f)} = \varphi^{-1}(U_f)$

$\exists\ V_x \ni x$ s.t. $\alpha(f)\big|_{V_x}$ is invertible.

$\Rightarrow$ $\exists$ covering $X_{\alpha(f)} = \bigcup_{i \in I} V_i$ $\forall i\ \exists\ h_i \in \mathcal{O}_X(V_i)$

s.t. $\alpha(f)\big|_{V_i} \cdot h_i = 1$

Claim: the h_i glue together to an inverse for $\alpha(f)\big|_{X_{\alpha(f)}}$.

$$h_i\big|_{V_i \cap V_j} \cdot \alpha(f)\big|_{V_i \cap V_j} = 1$$

and $$h_j\big|_{V_i \cap V_j}\ \alpha(f)\big|_{V_i \cap V_j} = 1$$

$\Longrightarrow$ uniqueness of inverses $$h_i\big|_{V_i \cap V_j} = h_j\big|_{V_i \cap V_j}$$

$\Rightarrow \quad \exists!\ h \in \mathcal{O}_X(X_{\alpha(f)})$ s.t. $h\,\alpha(f)|_{X_{\alpha(f)}} = 1.$

$\Rightarrow \quad \alpha(f)|_{X_{\alpha(f)} = \varphi^{-1}(U_f)}$ is invertible

$\Rightarrow$ we have a factorization (unique)

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{O}_X(X) \arrow[d] \\ A[f^{-1}] \arrow[r, "\alpha_f"'] & \mathcal{O}_X(X_{\alpha(f)}) \end{tikzcd}$$

To define $\varphi^{\#}$ for arbitrary open sets, we pass to the inverse limit: $\mathcal{O}_A(U) = \varprojlim_{U_f \subset U} \mathcal{O}_A(U_f)$

In order for this to work, we need compatibility with the restriction morphisms, i.e., we need to show that whenever $U_g \subset U_f$, we have a commutative diagram:

$$\begin{tikzcd}
f, g \in A \arrow[r, "\alpha"] \arrow[d] \arrow[dd, bend right] & \mathcal{O}_X(X) \arrow[d] \arrow[dd, bend left] \\
\frac{f}{1}, \frac{g}{1} \in A[f^{-1}] \arrow[r, "\alpha_f"] \arrow[d] & \mathcal{O}_X(X_{\alpha(f)}) \arrow[d] \\
\boxed{A[f^{-1}]\left[\left(\frac{g}{1}\right)^{-1}\right] \overset{\text{universal property}}{=} A[g^{-1}]} \arrow[r, "\alpha_g"] & \mathcal{O}_X(X_{\alpha(g)})
\end{tikzcd}$$

$h \longmapsto \alpha(h)$

$\frac{f}{h} \longmapsto \alpha(f)\alpha(h)^{-1}$

follows from the universal property of localizations. or the explicit description of α_f, α_g.

<u>Recall</u>: what it means for U_f to contain U_g:

$$U_g \subset U_f$$

$$\{p \not\ni g\} \subset \{p \not\ni f\}$$

$$\Longleftrightarrow \{p \ni g\} \supset \{p \ni f\} \Longleftrightarrow \sqrt{(f)} \ni g$$

$$\Longleftrightarrow \exists\, n > 0,\ \exists\, a \in A \text{ s.t. } g^n = af$$

So we have $\varphi^{\#} : \mathcal{O}_{\operatorname{Spec} A} \longrightarrow \varphi_{*}\mathcal{O}_X$

$\forall\ x \in X \qquad \varphi_x : \mathcal{O}_{A,x} \longrightarrow \mathcal{O}_{X,x}$

recall $\varphi(x) = \alpha^{-1}(m_x) \subset A$

$\in \operatorname{Spec} A$

$$\begin{tikzcd}
\varphi(x) = \alpha^{-1}(m_x) \\
\mathfrak{p} \subset A \arrow[r] \arrow[d] & \mathcal{O}_X(X) \arrow[d] \\
A_{\mathfrak{p}} = \mathcal{O}_{A,x} \arrow[r, "\varphi_x"] & \mathcal{O}_{X,x}
\end{tikzcd}$$

$\cup$

$\mathfrak{p}A_{\mathfrak{p}}$ maximal ideal

$\Rightarrow \boxed{\varphi_x^{-1}(m_x) = \mathfrak{p}A_{\mathfrak{p}}}$

$\Rightarrow \varphi : X \longrightarrow \operatorname{Spec} A$ is a morphism of locally ringed space.

__Injectivity:__ If two morphisms $\varphi, \psi : X \longrightarrow \operatorname{Spec} A$ induce the same map $\alpha : A \longrightarrow \mathcal{O}_X(X)$, then $\varphi = \psi$.

$$\varphi^{\#}(\operatorname{Spec} A) = \psi^{\#}(\operatorname{Spec} A) = \alpha : A \longrightarrow \mathcal{O}_X(X)$$

For $x \in X$, put $\mathfrak{p} := \varphi(x)$, $\mathfrak{q} := \psi(x)$

$$\begin{tikzcd}
 & A \arrow[r] \arrow[dl, "ev_{\mathfrak{p}}"'] \arrow[dd, "ev_{\mathfrak{q}}"] \arrow[dr, "ev_x \circ \alpha"] & \mathcal{O}_X(X) \arrow[d, "ev_x"] \\
A_{\mathfrak{p}} \arrow[rr, "\varphi_x"] & & \mathcal{O}_{X,x} \\
 & A_{\mathfrak{q}} \arrow[ur, "\psi_x"'] &
\end{tikzcd}$$

we know:

$$\varphi_x^{-1}(\mathfrak{m}_x) = \mathfrak{p} A_{\mathfrak{p}}$$

$$\psi_x^{-1}(\mathfrak{m}_x) = \mathfrak{q} A_{\mathfrak{q}}$$

The diagrams commute: $ev_x \circ \alpha = \varphi_x \circ ev_{\mathfrak{p}} = \psi_x \circ ev_{\mathfrak{q}}$

$$\Longrightarrow \quad (ev_x \circ \alpha)^{-1}(\mathfrak{m}_x) = (\varphi_x \circ ev_{\mathfrak{p}})^{-1}(\mathfrak{m}_x) = (\psi_x \circ ev_{\mathfrak{q}})^{-1}(\mathfrak{m}_x)$$

$$(\varphi_x \circ ev_{\mathfrak{p}})^{-1}(\mathfrak{m}_x) = ev_{\mathfrak{p}}^{-1}(\varphi_x^{-1}(\mathfrak{m}_x)) = ev_{\mathfrak{p}}^{-1}(\mathfrak{p} A_{\mathfrak{p}}) = \mathfrak{p}$$

Similarly $(\varphi_x \circ ev_{\mathfrak{q}})^{-1}(m_x) = \mathfrak{q}$

$\Rightarrow \quad \mathfrak{p} = \mathfrak{q}$, i.e., $\quad \varphi(x) = \psi(x) \quad \forall\, x \in X$.

So $\varphi = \psi : X \longrightarrow \operatorname{Spec} A$ as maps of spaces.

The maps $\varphi^{\#}$ and $\psi^{\#}$ are also equal on all basic open sets U_f, as maps $A[f^{-1}] \longrightarrow \mathcal{O}_X(X_{\alpha(f)})$ because they are obtained by localizing α. $\square$

We generalized affine varieties: $\quad Y \subset k^n$, $\quad Y = V(I)$, $\quad I \subset k[y_1, \dots, y_n] = A$

$$A(Y) = A/I(Y)$$

$Y \longleftrightarrow$ scheme $\operatorname{Spec} A(Y)$

What do we do about projective varieties? We make everything homogeneous!