

Theorem: (II. 8.15) X separated of finite type over an algebraically closed field k . Then $\Omega'_{X/k}$ is a locally free sheaf of rank equal to the dimension of X if and only if X is a nonsingular variety / k (assume all irreducible components of X have the same dimension, i.e., X is of pure dimension). The same statement is true if k is not algebraically closed, provided that X is a variety.

One of the main ingredients in the proof is

Lemma (Prop. II. 8.7) Let R be a noetherian local ring with maximal ideal \mathfrak{m} and containing a field k isomorphic to its residue field R/\mathfrak{m} . $(k \hookrightarrow R \twoheadrightarrow R/\mathfrak{m})$. \cong

Then the map $\delta: \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{R/k}^1 \otimes_R k$
 from the sequence $\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{R/k}^1 \otimes_R k \longrightarrow \Omega_{k/k}^1 \xrightarrow{=} 0$
 obtained from $k \hookrightarrow R \twoheadrightarrow R/\mathfrak{m} \cong k$
 $(A \longrightarrow B \longrightarrow C = B/I)$
 is an isomorphism.

Embedded tangent spaces: (link to "familiar" things)

X a scheme of finite type k alg. closed.

$x \in X$ a closed point, $U = \text{Spec } A$ an affine open neighborhood of x .

$$\Rightarrow \exists A \cong k[x_1, \dots, x_n] / (f_1, \dots, f_r)$$

Via this presentation, we view U as the closed subscheme of \mathbb{A}_k^n with ideal $I := (f_1, \dots, f_r)$ from the tower of rings $k \longrightarrow k[x_1, \dots, x_n] \twoheadrightarrow A$

we obtain the exact sequence:

$$\begin{array}{ccccc}
 \widetilde{I/I^2} & \xrightarrow{\delta} & \widetilde{\Omega^1_{k[x_1, \dots, x_n]/k}} \otimes A & \longrightarrow & \widetilde{\Omega^1_{A/k}} \longrightarrow 0 \\
 \parallel & & \parallel & & \parallel \\
 \widetilde{I/I^2} & & \overset{\rho^*}{\Omega^1_{\mathbb{A}_k^n/k}} & & \Omega^1_{U/k}
 \end{array}$$

After a translation, we can assume $x \mapsto (0, \dots, 0) \in \mathbb{A}^n$
 i.e., $x \mapsto$ maximal ideal $(x_1, \dots, x_n) = \mathfrak{m}_x$

From the lemma we have $\mathfrak{m}_{\mathbb{A}^n, x} / \mathfrak{m}_{\mathbb{A}^n, x}^2 \xrightarrow{\cong} \Omega^1_{\mathbb{A}^n/k, x} \otimes k \oplus \mathbb{O}_{\mathbb{A}^n, x}$

We know that $\Omega^1_{A^n/k} = \mathbb{C}_{A^n} dx_1 \oplus \dots \oplus \mathbb{C}_{A^n} dx_n$

$$\Rightarrow \mathcal{M}_{A^n, x} / \mathcal{M}_{A^n, x}^2 \xrightarrow{\cong} \left(\mathbb{C}_{A^n, x} dx_1 \oplus \dots \oplus \mathbb{C}_{A^n, x} dx_n \right) \otimes_{\mathbb{C}_{A^n, x}} k$$

||

$$k dx_1 \oplus \dots \oplus k dx_n$$

So we can identify the Zariski cotangent space $(T_x A^n)^*$ with the k -vector space with basis dx_1, \dots, dx_n .

$$\begin{array}{c} \text{"} \\ \mathcal{M}_{A^n, x} / \mathcal{M}_{A^n, x}^2 \end{array}$$

Modding out by the equations of V :

$$\Omega^1_{A/k} = \Omega^1_{A^n/k} / (df_1, \dots, df_r)$$

\Rightarrow The Zariski cotangent space to V at x can be identified with the quotient of $k dx_1 \oplus \dots \oplus k dx_n$ by $\mathfrak{S}^1_{A^n, x}$

the subspace generated by the images of

$$df_1, \dots, df_r \text{ in } \mathcal{O}_{A,x}/\mathfrak{m}_x \cong k$$

$$\begin{array}{c} \uparrow \\ k[x_1, \dots, x_n] / (x_1, \dots, x_n) \cong k \end{array}$$

(i.e., $df_1(0, \dots, 0), \dots, df_r(0, \dots, 0)$)

$$df_i(0, \dots, 0) = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(0, \dots, 0) dx_j$$

The scheme (U, \mathcal{O}_U) is nonsingular at x iff $\dim_k \mathfrak{m}_{U,x}/\mathfrak{m}_{U,x}^2$

$$= \dim A =: \dim U =: d$$

\Leftrightarrow the Jacobian matrix

$$\left(\frac{\partial f_i}{\partial x_j}(0, \dots, 0) \right)_{\substack{1 \leq i \leq r \\ 1 \leq j \leq n}}$$

has rank $n-d$.

The projective case: Let $X \subset \mathbb{P}_k^n$ be a

quasi-projective scheme / k alg. closed.

By this we mean X is an open subscheme of a closed subscheme Y of \mathbb{P}_k^n . Let I_Y be the homogeneous ideal of Y : $I_Y := \bigoplus_{n \geq 0} T(\mathbb{P}_k^n, \mathcal{I}_Y(n))$

Choose homogeneous generators P_1, \dots, P_r of respective degrees d_1, \dots, d_r for I_Y .

Let $x \in X$ be a closed point. We can assume, after a change of coordinates that x is the origin of $U_0 = D_+(X_0)$

i.e., $w_x = \left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0} \right)$ in U_0 . $\cong \mathbb{A}_k^n$

Then $I_{Y \cap U_0} = \left(\frac{P_1}{X_0^{d_1}}, \dots, \frac{P_r}{X_0^{d_r}} \right)$

Since X and Y have the same local rings and the same Zariski cotangent or tangent spaces at x , we replace X with Y and assume X is projective.

We apply the general result we had before:

$\mathcal{M}_{X,x} / \mathcal{M}_{X,x}^2$ can be identified with the quotient

of $k d\left(\frac{X_1}{X_0}\right) \oplus \dots \oplus k d\left(\frac{X_n}{X_0}\right)$ by the subspace generated by $d\left(\frac{P_1}{X_0^{d_1}}\right)(0, \dots, 0), \dots, d\left(\frac{P_r}{X_0^{d_r}}\right)(0, \dots, 0)$

By the Leibnitz rule

$$(*) \quad d\left(\frac{P_i}{X_0^{d_i}}\right) = \frac{X_0 dP_i - P_i dX_0}{X_0^{d_i+1}} \stackrel{\text{on } X}{=} \frac{dP_i}{X_0^{d_i}}$$

$$dP_i = \sum_{j=0}^n \frac{\partial P_i}{\partial X_j} dX_j$$

$$(*) \Rightarrow \text{the rank of } \left(\begin{array}{c} \frac{\partial \left(\frac{P_i}{X_0^{d_i}} \right)}{\partial \left(\frac{X_j}{X_0} \right)} (0, \dots, 0) \\ \vdots \\ \frac{\partial \left(\frac{P_n}{X_0^{d_n}} \right)}{\partial \left(\frac{X_j}{X_0} \right)} (0, \dots, 0) \end{array} \right)_{\substack{1 \leq i \leq r \\ 1 \leq j \leq n}}$$

$$\text{is equal to the rank of } \left(\begin{array}{c} \frac{\partial P_i}{\partial X_j} (0, \dots, 0) \\ \vdots \\ \frac{\partial P_r}{\partial X_j} (0, \dots, 0) \end{array} \right)_{\substack{1 \leq i \leq r \\ 0 \leq j \leq n}}$$

$$\Rightarrow \text{the vector space } \frac{k dX_0 \oplus \dots \oplus k dX_n}{(dP_1(0, \dots, 0), \dots, dP_r(0, \dots, 0))}$$

$$\text{has dimension 1 more than } \frac{k d\left(\frac{X_1}{X_0}\right) \oplus \dots \oplus k d\left(\frac{X_n}{X_0}\right)}{(d\left(\frac{P_1}{X_0^{d_1}}\right)(0, \dots, 0), \dots, d\left(\frac{P_r}{X_0^{d_r}}\right)(0, \dots, 0))}$$

$$\text{So } X \text{ is nonsingular at } x \Leftrightarrow \left(\begin{array}{c} \frac{\partial P_i}{\partial X_j} (0, \dots, 0) \\ \vdots \\ \frac{\partial P_r}{\partial X_j} (0, \dots, 0) \end{array} \right) \text{ has rank } n-d.$$

Zariski tangent spaces:

In the general case,

$$k(d_{f_1}|_{(x)} + \dots + d_{f_r}|_{(x)}) \rightarrow k dx_1 \oplus \dots \oplus k dx_n \rightarrow \mathfrak{m}_{U,x} / \mathfrak{m}_{U,x}^2 \rightarrow 0$$

dualize:

$$(k d_{f_1}|_{(x)} + \dots + k d_{f_r}|_{(x)})^* \leftarrow T_x \mathbb{A}^n \leftarrow T_x U \leftarrow 0$$

\Rightarrow equations for $T_x U \subset T_x \mathbb{A}^n$ are given

(as linear forms on k^n) by $d_{f_1}(x), \dots, d_{f_r}(x)$:

$dx_i : k^n \rightarrow k$ linear forms

$$(a_1, \dots, a_n) \mapsto a_i$$

$$df_i(x) (a_1, \dots, a_n) = \left(\sum_{j=1}^n \frac{\partial f_i(x)}{\partial x_j} dx_j \right) (a_1, \dots, a_n)$$

$$= \sum_{j=1}^n \frac{\partial f_i(x)}{\partial x_j} dx_j (a_1, \dots, a_n)$$

$$= \sum_{j=1}^n \frac{\partial f_i(x)}{\partial x_j} a_j$$

$T_x U \subseteq T_x \mathbb{A}^n$ is the subspace with equations the above linear forms.

In the projective case, the Zariski tangent space to the affine cone over X at any point y above x is the subspace of $k^{n+1} = T_y \mathbb{A}^{n+1} = (k dx_0 \oplus \dots \oplus k dx_n)^*$ with equations $dP_1(y), \dots, dP_n(y)$.