# The Euler sequence:

For any ring $A$, we have an exact sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n_A/\operatorname{Spec} A} \longrightarrow \mathcal{O}_{\mathbb{P}^n_A}(-1)^{\oplus(n+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^n_A} \longrightarrow 0$$

**Proof:** $\mathbb{P}^n_A = \operatorname{Proj} A[X_0, \dots, X_n]$ $\quad S := A[X_0, \dots, X_n]$

The sections $X_0, \dots, X_n \in \Gamma(\mathcal{O}_{\mathbb{P}^n_A}(1))$ generate $\mathcal{O}_{\mathbb{P}^n_A}(1)$:

on $U_i = D_+(X_i)$ $\quad \mathcal{O}_{\mathbb{P}^n_A}(1)\big|_{U_i} = \mathcal{O}_{U_i} X_i$

For each $i$, we have

$$X_i : \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)$$

If we add these maps, we obtain a surjective homomorphism

$$\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^n}(1)$$

Twisting this by $\mathcal{O}_{\mathbb{P}^n}(-1)$ we obtain

$$\Phi : \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^n}$$

<u>Claim:</u> the kernel of this map is naturally isomorphic to

$$\Omega^1_{\mathbb{P}^n_A/\operatorname{Spec} A}.$$

<u>proof of the claim:</u> Recall that $\mathcal{O}_{\mathbb{P}^n}(d) = \widetilde{S[d]}$

On each $U_i$: $\mathcal{O}_{\mathbb{P}^n}(d)\big|_{U_i} = \widetilde{S[d][X_i^{-1}]_0}$

The morphism $\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^n}$

on $U_i$ is obtained from

$$\varphi_i : S[-1][X_i^{-1}]_0^{\oplus n+1} \twoheadrightarrow S[X_i^{-1}]_0$$

by applying the $\sim$ functor.

Call $e_0, \ldots, e_n$ the basis $(1,0,\ldots,0), \ldots, (0,\ldots,1)$

of $S[-1][X_i^{-1}]_0$. Then $\varphi_i(\sum f_j e_j) = \sum f_j X_j$

Note that $\varphi_i\left(e_j - \frac{X_j}{X_i} e_i\right) = X_j - \frac{X_j}{X_i} X_i = 0$

Since $e_0 - \frac{X_0}{X_i} e_i, \ldots, e_n - \frac{X_n}{X_i} e_i, e_i$ is also

a basis of $S[-1][X_i^{-1}]_0^{\oplus(n+1)}$, we have that the

kernel of $\varphi_i$ is the free module $M_i$ with basis

$$e_0 - \frac{X_0}{X_i} e_i, \dots, e_n - \frac{X_n}{X_i} e_i \text{ over } S[X_i^{-1}]_0 .$$

We now define isomorphisms $\Omega^1_{\mathbb{P}^n_A}|_{U_i} \xrightarrow{\cong} \widetilde{M}_i$ which glue to a global isomorphism $\Omega^1_{\mathbb{P}^n} \longrightarrow \ker \Phi$.

$$\Omega^1_{\mathbb{P}^n}|_{U_i} \cong \Omega^1_{U_i} \qquad U_i = \operatorname{Spec} A\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right]$$

and $\Omega^1_{U_i}$ = free $\mathcal{O}_{U_i}$ module with basis

$$d\left(\frac{X_0}{X_i}\right), \dots, d\left(\frac{X_n}{X_i}\right)$$

define $\varphi_i : \Omega^1_{\mathbb{P}^n}|_{U_i} = \Omega^1_{U_i} \xrightarrow{\cong} \widetilde{M}_i$

$$d\left(\frac{X_j}{X_i}\right) \longmapsto \frac{X_i e_j - X_j e_i}{X_i^2}$$

The morphisms $\varphi_i$ glue together:

On $U_i \cap U_j$ : $\forall k$ $\quad \dfrac{X_k}{X_i} = \dfrac{X_k}{X_j} \cdot \dfrac{X_j}{X_i}$

hence $$d\left(\frac{X_k}{X_i}\right) = \frac{X_k}{X_j}\, d\left(\frac{X_j}{X_i}\right) + \frac{X_j}{X_i}\, d\left(\frac{X_k}{X_j}\right)$$

$$\Rightarrow \quad d\left(\frac{X_k}{X_i}\right) - \frac{X_k}{X_j}\, d\left(\frac{X_j}{X_i}\right) = \frac{X_j}{X_i}\, d\left(\frac{X_k}{X_j}\right)$$

apply $\varphi_i$ on the left, apply $\varphi_j$ on the right:

$$\varphi_i\left(d\left(\frac{X_k}{X_i}\right) - \frac{X_k}{X_j}\, d\left(\frac{X_j}{X_i}\right)\right) = \frac{X_i e_k - X_k e_i}{X_i^2} - \frac{X_k}{X_j}\, \frac{X_i e_j - X_j e_i}{X_i^2}$$

$$\varphi_j\left(\frac{X_j}{X_i}\, d\left(\frac{X_k}{X_j}\right)\right) = \frac{X_j}{X_i}\, \frac{X_j e_k - X_k e_j}{X_j^2}$$ //? YES

so $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$.

So the $\psi_i$ glue together to a global isomorphism.

$$\psi : \Omega^1_{\mathbb{P}^n} \longrightarrow \operatorname{Ker} \Phi.$$

$\square$

Back to the definition of $\Phi$:

We could have defined $\Phi$ on graded modules first:

$$S[-1]^{\oplus(n+1)} \longrightarrow S$$
$$(f_0, \dots, f_n) \longmapsto \sum_{i=0}^{n} f_i X_i$$

Define $M$ to be the kernel:

$$0 \longrightarrow M \longrightarrow S[-1]^{\oplus(n+1)} \longrightarrow S$$

$$\Rightarrow \quad \begin{tikzcd} 0 \arrow[r] & \tilde{M} \arrow[r] & \widetilde{S[-1]^{\oplus(n+1)}} \arrow[r] \arrow[d, phantom, "\parallel"] & \tilde{S} \arrow[d, phantom, "\parallel"] \\ 0 \arrow[r] & \tilde{M} \arrow[r] & \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \arrow[r, "\Phi"] & \mathcal{O}_{\mathbb{P}^n} \arrow[r] & 0 \end{tikzcd}$$

$$\Rightarrow \quad \tilde{M}|_{U_i} = \widetilde{M_i} \Rightarrow M_i = M[x_i^{-1}]_0$$