Given two morphisms of complexes $v^\cdot, w^\cdot : C^\cdot \to I^\cdot$ which are both extensions of u, we show that they are homotopic.

We put $h^{-1} = 0 : A = C^{-1} \longrightarrow 0$

Suppose we have constructed h^i for $i \le n : C^i \longrightarrow I^{i-1}$

s.t. $v^i - w^i = d^{i-1}h^i + h^{i+1}d^i$

we want h^{n+1} s.t. $v^n - w^n = dh^n + h^{n+1}d$

or $v^n - w^n - d^{n-1}h^n = h^{n+1}d^n$

$$
\begin{tikzcd}
C^{n-1} \arrow[d, "d^{n-1}"'] & I^{n-1} \arrow[d] \\
C^n \arrow[ur, "h^n"] \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "d^n"'] & I^n \arrow[d, "d^n"] \\
C^{n+1} \arrow[ur, dotted, "h^{n+1}"] \arrow[r] & I^{n+1}
\end{tikzcd}
$$

to show the existence of h^{n+1}, we need to show that $v^n - w^n - d^{n-1}h^n$ factors through C^{n+1} (or d^n)

$v^m - w^m - dh^m$ is zero on $\operatorname{im} d^{m-1}$ because

$$(v^m - w^m - dh^m)d = (v^m - w^m)d - d(h^m d)$$
$$= (v^m - w^m)d - d(v^{m-1} - w^{m-1} - dh^{m-1})$$
$$= 0$$

So $v^m - w^m - dh^m$ is zero on $\ker d^m = \operatorname{im} d^{m-1}$ and factors through $C^m / \ker d^m \hookrightarrow C^{m+1}$.

This factorization and hence also $v^m - w^m - dh^m$ extends to a morphism $h^{m+1} : C^{m+1} \longrightarrow I^m$.

$\square$

Properties of R^iF:

(1) $\forall i$ R^iF is an additive functor from $\mathcal{A}$ to $\mathcal{B}$.

(Can also show that $R^iF(A\oplus B) = R^iF(A)\oplus R^iF(B)$)

(2) There is a natural isomorphism $F \cong R^0F$

(3) If $I \in Ob(\mathcal{A})$ is injective, then $R^iF(I)=0$ $\forall i>0$

(4) For any short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

there are coboundary morphisms $\delta^i : R^iF(A'') \to R^{i+1}F(A')$

s.t.

$$\cdots \to R^iF(A') \to R^iF(A) \to R^iF(A'') \xrightarrow{\delta^i} R^{i+1}F(A') \to \cdots$$

is exact.

Furthermore, the δ^i are functorial, i.e., given a morphism

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0 \end{tikzcd}$$

the coboundary morphisms form commutative diagrams

$$\begin{tikzcd} R^iF(A'') \arrow[r, "\delta^i"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & R^{i+1}F(A') \arrow[d] \\ R^iF(B'') \arrow[r, "\delta^i"] & R^{i+1}F(B') \end{tikzcd}$$

Sketch of proof: To construct δ^i, choose injective resolutions $I'^{\bullet}$ and $I''^{\bullet}$ of A' and A'', use Lemma 2 above to construct morphisms fitting in a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I'^{\bullet} \arrow[r] & I'^{\bullet} \oplus I''^{\bullet} \arrow[r] & I''^{\bullet} \arrow[r] & 0
\end{tikzcd}$$

<u>Definition:</u> An object J is called acyclic for F if

$$R^iF\,(J) = 0 \quad \forall\, i > 0 .$$

<u>Remark:</u> One can use acyclic resolutions instead of injective resolutions to compute the right derived functors.

Sketch of proof: Given an acyclic resolution $A \longrightarrow C^{\cdot}$, let B^i be the image of C^i in C^{i+1} for $i \geq 0$ and put $B^{-1} := A$. The n-th cohomology of the complex $F(C^{\cdot})$ is then the cokernel of the map $F(C^{n-1}) \longrightarrow F(B^{n-1})$ for $n \geq 1$.

For $i \geq 0$ we have the short exact sequence

$$0 \to B^{i-1} \to C^i \to B^i \to 0$$

which gives $R^i F(A) \cong R^{i-1} F(B^0) \cong \cdots \cong R^1 F(B^{i-2})$ for $i \geq 1$

and $0 \to F(B^{i-2}) \to F(C^{i-1}) \to F(B^{i-1}) \to R^1 F(B^{i-2}) \to 0$

Hence $R^1 F(B^{i-2}) = \mathrm{coker}\left(F(C^{i-1}) \to F(B^{i-1})\right) = h^i(F(C^\bullet))$

Now we move towards defining cohomology of sheaves on schemes.

Lemma 3: The category of modules over a ring R with a unit (not necessarily commutative) has enough injectives.

Idea of proof: The R-module $R^+ := \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ is injective. Cyclic modules admit nonzero maps to R^+ and for any R-module A: $A \hookrightarrow \prod_{a \in A} R^+$ $\qquad Ra \xrightarrow{\neq 0} R^+$.

$\Rightarrow$ Lemma 4: For a ringed space $(X, \mathcal{O}_X)$, the category of sheaves of $\mathcal{O}_X$-modules has enough injectives.

<u>Idea of proof</u>: Given an $\mathcal{O}_X$-module $\mathcal{F}$, embed each stalk $\mathcal{F}_x$ $(x \in X)$ into an injective $\mathcal{O}_{X,x}$-module I_x. Then put $\mathcal{J} := \prod_{x \in X} (i_x)_* I_x$

$$i_x : \{x\} \hookrightarrow X$$

$\forall$ $\mathcal{O}_X$-module $\mathcal{G}$, we have $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{J})$

$$= \prod_{x \in X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, (i_x)_* I_x),$$

and $\forall$ $x \in X$, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, (i_x)_* I_x) = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x)$

$\Rightarrow$ <u>Corollary</u>: Apply this to $(X, \mathbb{Z}_X)$ where X is any topological space and $\mathbb{Z}_X$ is the locally constant sheaf.

We obtain that the category of sheaves of abelian groups on X has enough injectives.

<u>Fact</u>: The category of modules over a ring R has enough projectives: any module is the quotient of a free module which is projective.

However, the category of sheaves of abelian groups on a topological space X does <u>not</u> necessarily have enough projectives.

<u>Example</u>: Suppose X is a topological space s.t.

$\exists\, x \in X$ s.t. $\forall$ open $V \ni x$, $\exists$ an open $U \ni x$
closed point

s.t. $U \subsetneq V$ and U is connected.

For such U, V, let $\mathbb{Z}_{X,U} := i_! \mathbb{Z}_U$ be the extension by 0 of the constant sheaf $\mathbb{Z}_U$, i.e.,

$\mathbb{Z}_{X,U}(W) = \mathbb{Z}_U(W)$ if $W \subset U$, and $\mathbb{Z}_{X,U}(W) = 0$, if $W \not\subset U$.

Let $\mathbb{Z}_x$ be the skyscraper sheaf supported at x with stalk $\mathbb{Z}$, i.e., $\mathbb{Z}_x(W) = \mathbb{Z}$, if $x \in W$, and $\mathbb{Z}_x(W) = 0$ if $x \notin W$.

Note that there is a surjective morphism

$$\mathbb{Z}_{X,U} \twoheadrightarrow \mathbb{Z}_x$$

<u>Claim:</u> $\mathbb{Z}_x$ is NOT the quotient of a projective sheaf.

Otherwise if $\mathcal{P} \twoheadrightarrow \mathbb{Z}_x$ with $\mathcal{P}$ projective, then $\exists$ lift

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[rr, bend right] & \mathbb{Z}_{X,U} \arrow[r] & \mathbb{Z}_x \end{tikzcd}$$

We have $\mathbb{Z}_{X,U}(V) = 0$ because $V \not\subset U$

$\Rightarrow$ the map $\mathcal{F}(V) \longrightarrow \mathbb{Z}_x(V)$ which factors through $\mathbb{Z}_{X,U}(V)$ is 0.

If we fix x and $\mathcal{F}$, we can change U and V as above. This means that $\forall$ $V \ni x$ open $\mathcal{F}(V) \to \mathbb{Z}_x(V)$ is 0. $\Rightarrow$ the map on stalks $\mathcal{F}_x \longrightarrow \mathbb{Z}_x$ is 0: contradiction. $\square$