Definition 1: For a topological space X, the cohomology functors $H^i(X, \cdot): \mathcal{A}b(X) \longrightarrow \mathcal{A}b$ are the right-derived functors of the global sections functor $\Gamma(\cdot): \mathcal{A}b(X) \to \mathcal{A}b$; i.e.,

$$H^i(X, \mathcal{F}) = R^i\Gamma(\mathcal{F}).$$

In particular, $H^0(X, \mathcal{F}) = \Gamma(\mathcal{F})$.

Notation: From now on, we will use $H^0(X, \mathcal{F})$ to denote the global sections of $\mathcal{F}$.

Definition 2: A sheaf $\mathcal{F}$ on a topological space X is called flasque if, $\forall\, U \subset X$ open, the restriction map

$H^0(X, \mathcal{F}) \longrightarrow H^0(U, \mathcal{F})$ is surjective.

Lemma 1: On a ringed space $(X, \mathcal{O}_X)$, injective sheaves of $\mathcal{O}_X$-modules are flasque.

<u>Proof</u>: Suppose $\mathcal{I}$ is an injective sheaf of $\mathcal{O}_X$-modules.
For any open set $U \subset X$, we want to show

$$\begin{array}{ccc} H^0(X, \mathcal{I}) & \twoheadrightarrow & H^0(U, \mathcal{I}). \\ \| & & \| \\ \mathrm{Hom}(\mathcal{O}_X, \mathcal{I}) & & \mathrm{Hom}(\mathcal{O}_U, i^{-1}\mathcal{I}) \\ & & \| \\ & & \mathrm{Hom}(i_! \mathcal{O}_U, \mathcal{I}) \end{array}$$

note: since U is open, $i^{-1}\mathcal{I} = \mathcal{I}|_U$

$i : U \hookrightarrow X$

where $i_! \mathcal{O}_U$ is the extension of $\mathcal{O}_U$ to X by 0:

$$i_! \mathcal{O}_U(V) = \mathcal{O}_X(V) = \mathcal{O}_U(V) \text{ if } V \subset U$$
$$= 0 \quad \text{otherwise.}$$

$j : X \setminus U \hookrightarrow X$

Now (see Ex II.1.19) we have $0 \to i_! \mathcal{O}_U \to \mathcal{O}_X \to j_*(j^{-1}\mathcal{O}_X) \to 0$

$\mathrm{Hom}(\cdot, \mathcal{I})$ is exact $\Rightarrow \mathrm{Hom}(\mathcal{O}_X, \mathcal{I}) \twoheadrightarrow \mathrm{Hom}(i_! \mathcal{O}_U, \mathcal{I})$ $\square$

Lemma 2: $\mathcal{F}$ flasque sheaves are acyclic for the global sections functor.

Proof: Let $\mathcal{F}$ be a flasque sheaf on the topological space X. $\exists$ an injective sheaf $\mathcal{J}$ of abelian groups on X s.t. $\mathcal{F} \hookrightarrow \mathcal{J}$. Let $\mathcal{G}$ be the quotient:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{J} \longrightarrow \mathcal{G} \longrightarrow 0.$$

Ex: II.1.16: (and Lemma 1) $\mathcal{F}$ & $\mathcal{J}$ are flasque $\Rightarrow$ $\mathcal{G}$ flasque

$\mathcal{J}$ injective $\Rightarrow$ $\mathcal{J}$ acyclic. $\Rightarrow$ $H^i(X, \mathcal{J}) = 0 \;\forall i > 0$

$\mathcal{F}$ flasque (Ex. II.1.16) $\Rightarrow$ the sequence

$$0 \to H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{J}) \longrightarrow H^0(X, \mathcal{G}) \longrightarrow 0$$

is exact.

We have the long exact sequence of cohomology:

$$0 \to H^0(\mathcal{F}) \to H^0(\mathcal{I}) \to H^0(\mathcal{G}) \xrightarrow{0} \underset{0}{H^1(\mathcal{F})} \to \overset{0}{H^1(\mathcal{I})} \to H^1(\mathcal{G})$$
$$\to H^2(\mathcal{F}) \to \underset{0}{H^2(\mathcal{I})} \to H^2(\mathcal{G}) \to H^3(\mathcal{F}) \to \underset{0}{\cdots} \quad \cdots$$

$$\Rightarrow \quad H^1(\mathcal{F}) = 0$$
and $\quad H^i(\mathcal{G}) \xrightarrow{\cong} H^{i+1}(\mathcal{F}) \quad \forall \ i \geq 1$

by induction $\Rightarrow \ H^i(\mathcal{F}) = 0 \quad \forall \ \mathcal{F}$ flasque and $i > 0$. $\square$

<u>Corollary</u>: All cohomology groups on a ringed space $(X, \mathcal{O}_X)$ are $H^0(X, \mathcal{O}_X)$-modules.

<u>Proof</u>: Flasque sheaves are acyclic by Lemma 2. So we can use flasque resolutions to compute H^i. On any $(X, \mathcal{O}_X)$,

the injective sheaves of $\mathcal{O}_X$-modules are flasque,
so we can use resolutions by injective sheaves of $\mathcal{O}_X$-mod.
to compute H^i. $\Rightarrow H^i(X, \mathcal{F})$ are $H^0(X, \mathcal{O}_X)$-modules
$\forall$ $\mathcal{F}$ $\mathcal{O}_X$-mod. $\square$.

<u>Theorem:</u> Let X be a noetherian topological space of dimension n. Then, for any sheaf $\mathcal{F}$ of abelian groups on X and any $i > n$, $H^i(X, \mathcal{F}) = 0$

<u>Proof:</u> <u>Step 1:</u> Reduce to the case where X is irreducible by induction on the number of irreducible components:
If $Y \subset X$ is an irreducible component, put
$U := X \setminus Y$, $\bar{U}$ has one irreducible component less than X

(Ex. II. 1.19), we have the exact sequence

$$0 \to i_! i^{-1}\mathcal{F} \to \mathcal{F} \to j_* j^{-1}\mathcal{F} \to 0$$

where $i: U \hookrightarrow X$ $\quad j: Y \hookrightarrow X$

So the vanishing of $H^k(i_! i^{-1}\mathcal{F})$ and $H^k(j_* j^{-1}\mathcal{F})$

imply the vanishing of $H^k(\mathcal{F})$.

(1) $H^k(i_! i^{-1}\mathcal{F}) = H^k(\mu^{-1}(i_! i^{-1}\mathcal{F}))$ on $\bar{U}$, where $\mu: \bar{U} \hookrightarrow X$,

(2) $H^k(j_* j^{-1}\mathcal{F}) = H^k(j^{-1}\mathcal{F})$ on Y.

For (2) we have a Lemma:

<u>Lemma 3</u>: $\forall$ closed subset $Y \overset{j}{\hookrightarrow} X$, if $\mathcal{F}$ is a sheaf of abelian groups on Y, then $H^k(Y, \mathcal{F}) = H^k(X, j_*\mathcal{F})$.

Proof: If $\mathcal{J}^\cdot$ is a flasque resolution of $\mathcal{F}$ on Y, then $j_* \mathcal{J}^\cdot$ is a flasque resolution of $j_* \mathcal{F}$ on X.

and $\forall\, l$ $\quad H^0(Y \cap U, \mathcal{J}^l) = H^0(U, j_* \mathcal{J}^l)$

$U \subset X$ open $\quad H^0(Y, \mathcal{J}^l) = H^0(X, j_* \mathcal{J}^l)$

$$\Rightarrow \quad H^k(Y, \mathcal{F}) = H^k(X, j_* \mathcal{F}). \qquad \square$$

For (1), using Ex. II.1.19, the stalk $(i_! i^{-1}\mathcal{F})_x = 0 \quad \forall\, x \notin U$

this implies that the natural map $i_! i^{-1}\mathcal{F} \longrightarrow \mu_* \mu^{-1}(i_! i^{-1}\mathcal{F})$,

where $\mu : \bar{U} \hookrightarrow X$, is an isomorphism.

Hence, by Lemma 3, $H^l(i_! i^{-1}\mathcal{F}) = H^l(\mu^{-1}(i_! i^{-1}\mathcal{F})) \quad \forall\, l$.

$$\begin{array}{c} \| \\ H^l(\mu_* \mu^{-1}(i_! i^{-1}\mathcal{F})) \end{array}$$

Step 2: True for $X = \text{point} = \{x\}$.

In this case we have an equivalence of categories

$$\mathcal{A}b(X) \xrightarrow{H^0 = \text{stalk at } x.} \mathcal{A}b$$

The global sections functor is exact in this case.

$$\Rightarrow \quad H^i = 0 \quad \forall\ i > 0.$$

Step 3: Induction on dimension, and reduction to the case of a finitely generated sheaf.

Put $\quad B := \coprod_{\substack{U \subset X \\ \text{open}}} \mathcal{F}(U)$ Given a subset α of B,

we let $\mathcal{F}_\alpha \subset \mathcal{F}$ be the smallest subsheaf of ab. gps of $\mathcal{F}$ containing α: this is the subsheaf of $\mathcal{F}$ generated by α. If $\mathcal{F}_\alpha$ can be generated by a finite subset of B, we say it is finitely generated.

The set A of all finitely generated subsheaves of $\mathcal{F}$ is a directed set, and

$$\mathcal{F} = \varinjlim_{\alpha \in A} \mathcal{F}_\alpha$$

Note: The subsheaf generated by $s \in \mathcal{F}(U)$ is the image of $i_! \mathbb{Z}_U \xrightarrow{s} \mathcal{F}$ sending s to itself.

$\forall \alpha \in A$ we have $\mathcal{F}_\alpha \hookrightarrow \mathcal{F} = \varinjlim \mathcal{F}_\alpha$

$$\Rightarrow \quad H^k(\mathcal{F}_\alpha) \longrightarrow H^k(\mathcal{F}) = H^k(\varinjlim \mathcal{F}_\alpha)$$

$$\Rightarrow \quad \varinjlim H^k(\mathcal{F}_\alpha) \longrightarrow H^k(\mathcal{F}) = H^k(\varinjlim \mathcal{F}_\alpha)$$

<u>Ex. II.1.11</u> this map is an isom. when $i = 0$.

One can show this implies the map is an isom $\forall i$.

$\Rightarrow$ Enough to show the theorem for finitely generated sheaves.

Now assume $\mathcal{F}$ is a finitely generated sheaf.
By induction on the number of generators, we can assume it is generated by one element, say $s \in H^0(U, \mathcal{F})$ for some open set $U \overset{i}{\hookrightarrow} X$.

<u>Note</u>: If $\mathcal{F}$ is generated by $s_i \in \mathcal{F}(U_i)$, $1 \leq i \leq n$, then $\mathcal{F}/\text{image of } i_{n!}\mathbb{Z}_{U_n} s_n$ is generated by the images of $s_1, \dots, s_{n-1}$.

This means we have a surjection

$$i_!\mathbb{Z}_U \twoheadrightarrow \mathcal{F}.$$
$$1 \longmapsto s$$

We can complete this to an exact sequence:

$$0 \to \mathcal{R} \to i_!\mathbb{Z}_U \to \mathcal{F} \to 0.$$