The functors $\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \cdot)$ and $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \cdot)$ are defined to be the right derived functors of $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \cdot)$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \cdot)$ respectively.

<u>Properties of Ext^i and $\mathcal{E}xt^i$:</u>

(1) For any $\mathcal{O}_X$-module $\mathcal{G}$:

$\mathcal{E}xt^0(\mathcal{O}_X, \mathcal{G}) \cong \mathcal{G}$, $\mathcal{E}xt^i(\mathcal{O}_X, \mathcal{G}) = 0 \ \forall i > 0$

$\operatorname{Ext}^0(\mathcal{O}_X, \mathcal{G}) \cong \operatorname{Hom}(\mathcal{O}_X, \mathcal{G}) = H^0(X, \mathcal{G})$.

$\operatorname{Ext}^i(\mathcal{O}_X, \mathcal{G}) \cong H^i(X, \mathcal{G}) \ \forall i > 0$

(2) Short exact sequences of sheaves give rise to long exact sequences of exterior sheaves and groups.

(3) $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})$ and $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})$ can be computed using left resolutions of $\mathcal{F}$ by locally free sheaves of finite rank and applying the functors $\mathrm{Hom}_{\mathcal{O}_X}(\cdot, \mathcal{G})$ and $\mathcal{H}om_{\mathcal{O}_X}(\cdot, \mathcal{G})$ when such resolutions exist (for instance if $\mathcal{F}$ is coherent and X is quasi projective over a noetherian ring).

(4) If $\mathcal{L}$ is locally free of finite rank, then $\forall$ $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$:

$$\mathrm{Ext}^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \mathrm{Ext}^i(\mathcal{F}, \mathcal{L}^* \otimes \mathcal{G})$$

and

$$\mathcal{E}xt^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{L}^* \otimes \mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \otimes \mathcal{L}^*$$

(5) If X is a noetherian scheme and $\mathcal{F}$ is coherent, then $\forall\, x \in X$ and $\forall\, i$ and $\forall\, \mathcal{G}$ $\mathcal{O}_X$-module:

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

<u>Dualizing sheaves:</u>

<u>Def</u>: X proper scheme of dimension n over a field k. A dualizing sheaf for X is a coherent sheaf ω_X, together with a "trace" morphism

$$t : H^n(X, \omega_X) \longrightarrow k$$

s.t. $\forall$ coherent $\mathcal{F}$, the natural pairing

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X)$$

followed by t is a perfect pairing, i.e., it gives an isomorphism

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \cong H^n(X, \mathcal{F})^*.$$

Facts:

(1) Dualizing sheaves do not always exist, but are unique when they do.

(2) If X is projective, then

$$\omega_X \cong \mathcal{E}xt^r_{\mathbb{P}^m}(\mathcal{O}_X, \Omega^m_{\mathbb{P}^m})$$

where $X \hookrightarrow \mathbb{P}^m$, $r = m-n$ is the codimension of X in $\mathbb{P}^m$, $\Omega^m_{\mathbb{P}^m} := \Lambda^m \Omega^1_{\mathbb{P}^m}$

(3) In particular, apply (2) to $X = \mathbb{P}^n$:

$$\omega_{\mathbb{P}^n} \cong \mathcal{E}xt^0_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}, \Omega^n_{\mathbb{P}^n})$$
$$\cong \mathcal{H}om_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}, \Omega^n_{\mathbb{P}^n}) \cong \Omega^n_{\mathbb{P}^n}$$

Recall the Euler sequence:

$$0 \to \Omega^1_{\mathbb{P}^n}(1) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow 0$$

$$H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}$$

twist back by -1:

$$0 \to \Omega^1_{\mathbb{P}^n} \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \to 0$$

Take top exterior powers (past homework):

$$\Lambda^{n+1}\left(H^0(\mathcal{O}_{\mathbb{P}^n}(1))\otimes\mathcal{O}_{\mathbb{P}^n}(-1)\right)\cong\Lambda^n\Omega^1_{\mathbb{P}^n}\otimes\mathcal{O}_{\mathbb{P}^n}$$

$$\Lambda^{n+1}\left(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)}\right)\cong\Omega^n_{\mathbb{P}^n}$$

or $$\mathcal{O}_{\mathbb{P}^n}(-1)^{\otimes(n+1)}\cong\Omega^n_{\mathbb{P}^n}$$

or $$\mathcal{O}_{\mathbb{P}^n}(-n-1)\cong\Omega^n_{\mathbb{P}^n}=\omega_{\mathbb{P}^n}$$

(4) More generally, if X is a nonsingular projective (irred.) variety over a field, then the dualizing sheaf of X is its canonical sheaf, i.e., the top exterior power of its sheaf of differentials Ω^1_X.

(5) Serre Duality:

Theorem: If X is Cohen-Macaulay and projective of pure dimension n (i.e., all irreducible components have dimension n), then, for all coherent sheaves $\mathcal{F}$ and all i, there are functorial isomorphisms:

$$\operatorname{Ext}^i(\mathcal{F}, \omega_X) \cong H^{n-i}(X, \mathcal{F})^*.$$

For all locally free sheaves of finite rank:

$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \omega_X \otimes \mathcal{F}^*)^*$$

(recall $\mathcal{F}^* := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$)

(6) If X is projective and a local complete intersection of codimension r in $\mathbb{P}^n$, then

$$\omega_X \cong \omega_{\mathbb{P}^n} \otimes \overset{r}{\Lambda}\left(\mathcal{I}_X/\mathcal{I}_X^2\right)^*.$$

Def: $\left(\mathcal{I}_X/\mathcal{I}_X^2\right)^*$ is the normal sheaf of X in $\mathbb{P}^n$

$\mathcal{I}_X/\mathcal{I}_X^2$ is the conormal sheaf.

Regular sequences and the Cohen-Macaulay condition:

Def: given a ring A and an A-module M, a sequence $a_1, \dots, a_r$ of elements of A is called regular for M if

(1) a_1 is not a zero divisor in M $\left(a_1 : M \hookrightarrow M\right)$

(2) $\forall\, i \geq 2$, a_i is not a zero divisor in $M/(a_1, \dots, a_{i-1})M$.

Def: If A is a local ring with maximal ideal $\mathfrak{m}$, the depth of M is the maximum length of a regular sequence $\{a_1, \dots, a_n\} \subset \mathfrak{m}$.

A noetherian local ring is called Cohen-Macaulay if its depth as a module over itself is equal to its dimension.

Facts: Regular local rings are Cohen-Macaulay and quotients of Cohen-Macaulay rings by ideals generated by regular sequences are Cohen-Macaulay.

Def: (1) A scheme is Cohen-Macaulay if all its local rings are Cohen-Macaulay.

(2) A scheme is a local complete intersection if $\forall\ x \in X$ $\exists$ affine neighborhood $U = \operatorname{Spec} A \ni x$.

s.t. $A = B/I$ where I can be generated

by a sequence $a_1, \dots, a_n \in I$ s.t.

$\forall y \in \operatorname{Spec} B$ $\quad (a_1)_y, \dots, (a_n)_y \in (\tilde{I})_y$

is a regular sequence.

and $\operatorname{Spec} B$ is a regular scheme.

<u>Def</u>: A not necessarily local ring is called Cohen-Macaulay if all its localizations at its prime ideals are Cohen-Macaulay.

<u>Fact</u>: If $a_1, \dots, a_n$ is a regular sequence and $I := (a_1, \dots, a_n) \subset A$, then I/I^2 is a free A-module of rank n and, for all d, the natural map

$$\mathrm{Sym}^d(I/I^2) \longrightarrow I^d/I^{d+1}$$

is an isomorphism.

To say that a noetherian local ring of dimension n is Cohen-Macaulay means that there exists a regular sequence $a_1, \dots, a_n$ s.t. the quotient $A/(a_1, \dots, a_n)$ has dimension zero.

(When the ring is regular, $\exists$ sequence s.t. the quotient is a field)

Given a scheme X and a point $x \in X$, let $U = \operatorname{Spec} A$ be an affine neighborhood of x and let $\mathfrak{p}$ be the prime ideal of A corresponding to x. If $\mathcal{O}_{X,x} = A_{\mathfrak{p}}$ is Cohen-Macaulay, then its maximal ideal contains a regular sequence $a_1, \dots, a_d$ with $d = \dim A_{\mathfrak{p}} = \operatorname{height} \mathfrak{p}$. After possibly localizing A (at the denominators of $a_1, \dots, a_d$) we can assume $a_1, \dots, a_d \in A$.

By Hauptidealsatz,
$$\dim A/\langle a_1, \dots, a_d \rangle = \dim A - d = \dim \overline{\{\mathfrak{p}\}} = \dim A/\mathfrak{p}$$

Hence $\operatorname{Spec} A/\mathfrak{p} = \overline{\{\mathfrak{p}\}}$ is an irreducible component of $\operatorname{Spec} A/\langle a_1, \dots, a_d \rangle = Z(a_1, \dots, a_d) = Z(a_1) \cap \dots \cap Z(a_d)$

After further localizing A, we can assume $\operatorname{Spec} A/\langle a_1, \dots, a_d \rangle$

is irreducible (each irreducible component is closed and after removing it we have a smaller neighborhood of x).

Hence the closure $\overline{\{p\}}$ is "set-theoretically" cut out by the d equations $a_1, \dots, a_d$.

Examples of Cohen-Macaulay rings:

(1) Noetherian local rings of dimension 0 are Cohen-Macaulay.
Note that Noetherian local rings of dimension 0 are Artin rings.
A ring is an Artin ring if it satisfies the descending chain condition for ideals.

(2) One-dimensional reduced noetherian rings are Cohen Macaulay.

(3) Two dimensional normal noetherian rings are Cohen-Macaulay

A ring is called normal if it is reduced and integrally closed in its total quotient ring.

(4) If A is a finitely generated Cohen-Macaulay algebra over a field k with an action of a finite group G, then the subring of invariants A^G is Cohen-Macaulay.

(5) Determinantal rings are Cohen-Macaulay.

A ring is called determinantal if it is a quotient

$$B = A/I$$ where A is a regular local ring and

I is the ideal generated by the $r\times r$ minors of a $p\times q$

matrix with coefficients in A s.t. the height of any minimal prime of I is the expected codimension $(p-r+1)(q-r+1)$.

The main example of such an ideal is the ideal of the locus of matrices of rank $< r$ in the space of all $p \times q$ matrices with entries in a field.

Curves: From now on, all schemes are over an algebraically closed field k.

Def: A curve $X/_k$ is an integral separated scheme of finite type over k, of dimension 1. We say X is complete if it is proper over k. We will prove some nice results about curves.