BULLETIN DE LA S. M. F.

E. IZADI

Second order theta divisors on Pryms

Bulletin de la S. M. F., tome 127, n^o 1 (1999), p. 1-23.

<http://www.numdam.org/item?id=BSMF_1999__127_1_1_0>

Bull. Soc. math. France,
127, 1999, p. 1–23.

SECOND ORDER THETA DIVISORS ON PRYMS

BY E. IZADI

ABSTRACT. — Van Geemen and van der Geer, Donagi, Beauville and Debarre proposed characterizations of the locus of jacobians which use the linear system of 2Θ-divisors. We give new evidence for these conjectures in the case of Prym varieties.

RÉSUMÉ. — DIVISEURS THÊTA DU SECOND ORDRE SUR LES VARIÉTÉS DE PRYM. Van Geemen et van der Geer, Donagi, Beauville et Debarre ont proposé des caractérisations du lieu des jacobiennes qui utilisent les diviseurs thêta du second ordre. Nous prouvons des résultats partiels en direction de ces conjectures dans le cas des variétés de Prym.

The *Schottky problem* is the problem of finding necessary and sufficient conditions for a principally polarized abelian variety (*ppav*) to be a product of jacobians of smooth curves.

Let (P, Ξ) be an *indecomposable* ppav of dimension $p \geq 4$ with Ξ a symmetric theta divisor on P. The elements of

$$\Gamma = H^0(P, 2\Xi)$$

are symmetric, hence their multiplicities at the origin are always even. Let $\Gamma_0 \subset \Gamma$ be the subvector space of Γ of sections with multiplicity at least 2 at the origin and let $\Gamma_{00} \subset \Gamma$ be the subvector space of sections with multiplicity at least 4 at the origin. Also let

$$|2\Xi|_{00} \subset |2\Xi|_0 \subset |2\Xi|$$

be the linear systems of divisors of zeros of elements of $\Gamma_{00} \subset \Gamma_0 \subset \Gamma$ respectively. It is well-known that the dimensions of Γ, Γ_0 and Γ_{00} are respectively 2^p, $2^p - 1$ and $2^p - 1 - \frac{1}{2}p(p+1)$ (see [Ig, p. 188, Lemma 11] and [GG, Prop. 1.1, p. 618]).

(*) Texte reçu le 16 septembre 1997, révisé le 22 juin 1998, accepté le 11 août 1998.
E. IZADI, Departement of Mathematics, Boyd Graduate Studies Research Center, University of Georgia, Athens, GA 30602-7403 (USA). Email: izadi@math.uga.edu.

Keywords: Schottky, theta, Prym, Prym-canonical, jacobian, translate.

AMS classification: 14H42, 14K25.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037–9484/1999/1/$ 5.00

We have the linear map

$$\tau_4 \colon \Gamma_{00} \longrightarrow H^0\big(\mathbb{P}T_0P, \mathcal{O}_{\mathbb{P}T_0P}(4)\big)$$

defined by sending a section $s \in \Gamma_{00}$ to the quartic term of its Taylor expansion at the origin. Let $\mathcal{Q}_{00}$ denote the linear subsystem of $|\mathcal{O}_{\mathbb{P}T_0P}(4)|$ which is the projectivization of the image of τ_4. Let V_{00} and $V_{\mathrm{inf},00}$ denote the base loci of $|2\Xi|_{00}$ and $\mathcal{Q}_{00}$ respectively. In [GG], van Geemen and van der Geer proposed a characterization of the locus of jacobians, made more precise by Donagi [Do1, p. 110], in the following form:

CONJECTURE 1.
1) *If* $(P,\Xi) = (JC,\Theta)$ *is the jacobian of a smooth curve* C *of genus* p, *then* V_{00} *is set-theoretically equal to the reduced surface*

$$C - C \colon = \{\mathcal{O}_C(s-t) \colon s,t \in C\}.$$

2) *If* (P,Ξ) *is (indecomposable and) not in the closure* $\mathcal{J}_p$ *of the locus of jacobians in the moduli space of ppav of dimension* p, *then* $V_{00} = \{0\}$ *set-theoretically.*

Beauville and Debarre proposed an infinitesimal version of Conjecture 1 (see [BD]):

CONJECTURE 2.
1) *If* $(P,\Xi) = (JC,\Theta)$ *is the jacobian of a smooth curve* C *of genus* p, *then* $V_{\mathrm{inf},00}$ *is, set-theoretically, the canonical image* κC *of* C *in* $|\omega_C|^* = \mathbb{P}T_0JC$ *where* ω_C *is the dualizing sheaf of* C *(note that* κC *is the projectivized tangent cone to* $C - C$ *at* 0).

2) *If* (P,Ξ) *is (indecomposable and) not in* $\mathcal{J}_p$, *then* $V_{\mathrm{inf},00}$ *is empty.*

The first parts of Conjectures 1 and 2 have been proved (each with one well-determined exception) by Welters [W1], the author [Iz1] and Beauville and Debarre [BD]. In [Iz1], we also determined the scheme-structures of the base loci for jacobians. In [Iz2, Thm 4, p. 95], we proved the second parts of Conjectures 1 and 2 in the case $p = 4$. Beauville, Debarre, Donagi and van der Geer proved part 2) of Conjecture 1 for intermediate jacobians of cubic threefolds and the Prym varieties of "even" étale double covers of smooth plane curves (see [BDDG]). Beauville and Debarre proved parts 2) of Conjectures 1 and 2 for certain ppav isogenous to a product of p elliptic curves (see [BD, p. 35–38]). By semi-continuity, Beauville and Debarre then deduce from their result that for a *general* ppav V_{00} is finite and $V_{\mathrm{inf},00}$ is empty.

Let $\mathcal{A}_p$ be the moduli space of ppav of dimension p. In [Iz2], we proved the second parts of Conjectures 1 and 2 for $p = 4$ by using the fact (proved in [Iz2, Thm 3.3, p. 111]) that an element of $\mathcal{A}_4 \setminus \mathcal{J}_4$ is always the Prym variety of an étale double cover of a *smooth* curve of genus 5. From now on we will suppose that (P, Ξ) is the Prym variety of an étale double cover of smooth curves

$$\pi : \widetilde{C} \to C$$

with C *non-hyperelliptic* of genus $g = p + 1$ (then (P, Ξ) is automatically indecomposable, see [M2, p. 344, Thm (d)]). There is a natural analogue of the surface $C - C$ for a Prym variety, namely, the reduced surface

$$\Sigma : = \Sigma(\pi : \widetilde{C} \to C) : = \left\{ \mathcal{O}_{\widetilde{C}}(s + t - \sigma s - \sigma t) : s, t \in \widetilde{C} \right\} \subset P \subset J\widetilde{C},$$

where $\sigma : \widetilde{C} \to \widetilde{C}$ is the involution of the cover $\pi : \widetilde{C} \to C$. Let $\epsilon : \widetilde{P} \to P$ be the blow up of P at 0 with exceptional divisor $\mathcal{E}$ and let $\widetilde{\Sigma}$ be the proper transform of Σ in $\widetilde{P}$. Let L be the linear system $|\epsilon^*(2\Xi) - 4\mathcal{E}|$ on $\widetilde{P}$. When $g = 5$ there is an involution λ acting on the moduli space of admissible double covers of stable curves of genus 5 such that a double cover $\widetilde{C} \to C$ and $\lambda(\widetilde{C} \to C)$ have the same Prym variety (see [Do2, P. 100] and [Iz2, p. 119 and 126]). Furthermore, for any fixed $(P, \Xi) \in \mathcal{A}_4 \setminus \mathcal{J}_4$, there is an étale double cover $\widetilde{C} \to C$ of a smooth curve C such that

$$(\widetilde{C}_\lambda \to C_\lambda) : = \lambda(\widetilde{C} \to C)$$

is also an étale double cover of a smooth curve C_λ (see [Iz2 p. 136]). In such a case, put

$$\Sigma_\lambda : = \Sigma(\widetilde{C}_\lambda \to C_\lambda)$$

and let $\widetilde{\Sigma}_\lambda$ be the proper transform of Σ_λ in $\widetilde{P}$. With these hypotheses, we proved in [Iz2] that (recall $g - 1 = p = 4$):

- there is exactly a pencil of elements of L containing $\widetilde{\Sigma}$ (see [Iz2, 5.7, p. 134 and 6.23, p. 148]);
- the base locus of this pencil is equal to $\widetilde{\Sigma} \cup \widetilde{\Sigma}_\lambda$ as a set, as a scheme if $(\widetilde{C} \to C) \neq (\widetilde{C}_\lambda \to C_\lambda)$ (follows from [Iz2, 5.7, p. 134]);
- the base locus of the restriction $L_{|\widetilde{\Sigma}}$ (and, similarly, $L_{|\widetilde{\Sigma}_\lambda}$) is empty (see [Iz2, p. 139 and p. 146–147]).

We generalize this third result to higher-dimensional Prym varieties and calculate the dimension of the linear subsystem of L consisting of elements containing $\widetilde{\Sigma}$.

In Section 4 we prove:

THEOREM 3. — *If C is non-trigonal of genus ≥ 5, then the base locus of $L_{|\widetilde{\Sigma}}$ is empty.*

For C trigonal, the support of the base locus of $L_{|\widetilde{\Sigma}}$ is determined in Proposition 4.8 below.

To prove the theorem, we use divisors in the linear system $|2\Xi|_{00}$ which are obtained as intersections with $P \subset J\widetilde{C}$ of translates of the theta divisor $\widetilde{\Theta}$ of $J\widetilde{C}$ (see Section 3 below). To our knowledge such divisors have not been used before in the litterature.

The Prym-canonical curve χC is the image of C in $|\omega_C \otimes \alpha|^*$ by the natural morphism $C \to |\omega_C \otimes \alpha|^*$ where α is the square-trivial invertible sheaf associated to the double cover $\pi : \widetilde{C} \to C$. Under the natural isomorphism $\mathbb{P}T_0 P \cong |\omega_C \otimes \alpha|^*$, the curve χC is the tangent cone to Σ at 0. Therefore the Theorem implies:

COROLLARY 4. — *Suppose that C is not trigonal, then:*

1) *the only base point of $|2\Xi|_{00}$ on Σ is 0;*

2) *the linear system $\mathcal{Q}_{00}$ has no base points on χC.*

Our approach leads us to consider the vector space

$$\Gamma'_{00} : = \{ s \in \Gamma_{00} : s_{|\Sigma} = 0 \}$$

with projectivization

$$|2\Xi|'_{00} : = \{ D \in |2\Xi|_{00} : D \supset \Sigma \}.$$

Then $|2\Xi|'_{00}$ can be identified with the linear subsystem of elements of L containing $\widetilde{\Sigma}$. Let $\mathcal{Q}'_{00}$ be the linear subsystem of $\mathcal{Q}_{00}$ consisting of quartic tangent cones at 0 to elements of $|2\Xi|'_{00}$. On a Prym variety, second order theta divisors which contain Σ can be thought of as natural generalizations of 2Θ-divisors on JC containing $C - C$. It is well-known (see [F, Thm 2.5, p. 120] and [GG, p. 625] or [W1, Prop. 4.8, p. 18]) that, on a jacobian (JC, Θ), a 2Θ-divisor contains $C - C$ if and only if it has multiplicity at least 4 at the origin. This does not generalize to Prym varities: Corollary 4 implies that $|2\Xi|'_{00}$ and $\mathcal{Q}'_{00}$ are proper linear subsystems of $|2\Xi|_{00}$ and $\mathcal{Q}_{00}$ respectively. More precisely, we prove (see Section 4):

PROPOSITION 5. — *The dimension of Γ'_{00} is $2^p - 2 - p(p-1)$. The codimension of $\mathcal{Q}'_{00}$ in $\mathcal{Q}_{00}$ is at least $g - 3$.*

Note that when $p = 4$, we have another proof of the result of [Iz2, p. 148]) saying that the dimension of $|2\Xi|'_{00}$ is 1. We pose:

Conjecture 6. — *Suppose that* (P,Ξ) *is not a jacobian. Then:*

1) *the base locus* V'_{00} *of* $|2\Xi|'_{00}$ *is* Σ *as a set if* $p \geq 6$;

2) *the base locus* $V'_{\inf,00}$ *of* $\mathcal{Q}'_{00}$ *is* χC *as a set if* $p \geq 6$.

By Corollary 4, this conjecture implies the second parts of Conjecture 1 and 2 for Prym varieties. We have the following evidence for this conjecture.

Results of Welters and Debarre (see [W2] and [De], see also Section 1 below) imply:

Proposition 7. — *The base locus* V'_{00} *is the set-theoretical union of* Σ *and, possibly, some curves and points for a general Prym variety of dimension* $p \geq 16$.

Combined with Corollary 4 this implies:

Corollary 8. — *If* (P,Ξ) *is a general Prym variety of dimension* ≥ 16, *then* V_{00} *has dimension* ≤ 1.

Results of Debarre imply (see Section 1):

Proposition 9. — *For a general Prym variety of dimension* $p \geq 8$, *the base locus* $V'_{\inf,00}$ *is set-theoretically equal to* χC.

Combined with Corollary 4 this implies (see Section 1 below):

Corollary 10. — *Part* 2) *of Conjecture* 2 *is true for general Prym varieties of dimension* $p = g - 1 \geq 8$.

We explain why we make Conjecture 6 only for $p \geq 6$ and only set-theoretically. Let $\mathcal{R}_g$ be the space parametrizing étale double covers $\widetilde{C} \to C$ where C is a smooth (non-hyperelliptic) curve of genus $g = p + 1$. The Prym map is the morphism $\mathcal{R}_g \to \mathcal{A}_p$ which to a double cover $\widetilde{C} \to C$ associates its Prym variety. Recall (see above) that in case $p = 4$, we proved in [Iz2, 5.7, p. 134] that $V'_{00} = \Sigma \cup \Sigma_\lambda$. Since the Prym map is generically injective for $g \geq 7$ (see [FS]), we expect that $V'_{00} = \Sigma$ as sets. Now an argument analogous to [Iz1, (2.9), p. 196] shows that, if $V'_{00} = \Sigma$ as sets, then V'_{00} is not reduced at 0 and hence is not equal to Σ as a scheme. If the morphism $\rho: \widetilde{C}^{(2)} \to \Sigma$, where $\widetilde{C}^{(2)}$ is the second symmetric power of $\widetilde{C}$, is birational, then, by analogy with the case of jacobians (see [Iz1]), we can expect V'_{00} to be reduced at the generic point of Σ. If, on the other hand, the morphism ρ is not birational, then, by a standard semi-continuity argument, the scheme V'_{00} is not reduced anywhere on Σ. Note that, using a refinement of a theorem of Martens by Mumford (see [ACGH, Thm 5.2, p. 193]), one can easily see that if $\widetilde{C}$ is neither bielliptic, trigonal, nor a smooth plane quintic, then ρ is birational. Similarly, we can expect $V'_{\mathrm{inf},00}$ to be equal to χC as schemes if $C \cong \chi C$ but $V'_{\mathrm{inf},00}$ will not be equal to χC as a scheme if the morphism $C \to \chi C$ is not (at least) birational.

On the other hand, Donagi and Smith proved (see [DS]) that the generic fibers of the Prym map have cardinality 27 when $g = 6$ and Donagi proved (see [Do2, Thm 4.1, p. 90]) that the Galois group of the Prym map is isomorphic to the Galois group of the 27 lines on a cubic surface. So there could be nontrivial automorphisms acting in the fibers of the Prym map. If there is an automorphism $\mu: \mathcal{R}_6 \to \mathcal{R}_6$ acting in the fibers of the Prym map, then, as in the case $p = 4$, the base locus V'_{00} could be the union of $\Sigma = \Sigma(\widetilde{C} \to C)$ and the surfaces $\Sigma(\mu(\widetilde{C} \to C)), \Sigma(\mu^2(\widetilde{C} \to C))$, *etc.*

Ackowledgements. — I wish to thank E. Colombo, C. Pauly and P. Pirola for helpful discussions and the referee for helpful comments.

Notation and conventions

All varieties and schemes are over the field $\mathbb{C}$ of complex numbers. All ppav are indecomposable and all curves are smooth, complete, irreducible and non-hyperelliptic.

For any section s of an invertible sheaf $\mathcal{L}$ on a variety X, denote by $Z(s)$ the divisor of zeros of s. Let $h^i(X, \mathcal{L})$ denote the dimension of $H^i(X, \mathcal{L})$.

For a divisor D on C (resp. $\widetilde{C}$), denote by $\langle D \rangle$ the span of D in the canonical space of C (resp. $\widetilde{C}$).

For any subset Y of a group G and any element $a \in G$, denote by Y_a the translate of Y by a.

1. Preliminaries

Let $\pi: \widetilde{C} \to C$ be an étale double cover of a smooth (non-hyperelliptic) curve C of genus g. Let α be the point of order 2 in $\mathrm{Pic}^0 C$ associated to the double cover π so that we have

$$\pi_* \mathcal{O}_{\widetilde{C}} \cong \mathcal{O}_C \oplus \alpha.$$

Choose an element β of $\mathrm{Pic}^0 C$ such that $\beta^{\otimes 2} \cong \alpha$ and a theta-characteristic κ on C such that $h^0(C, \kappa)$ and $h^0(\widetilde{C}, \pi^*(\kappa \otimes \beta))$ are even. Symmetric principal polarizations on $J\widetilde{C} = \mathrm{Pic}^0 \widetilde{C}$ and $JC = \mathrm{Pic}^0 C$ can be defined as the reduced divisors $\widetilde{\Theta}: = \widetilde{\Theta}'_{(\kappa\otimes\beta)^{-1}}$ and $\Theta: = \Theta'_{\kappa^{-1}}$ where

$$\widetilde{\Theta}': = \{D \in \operatorname{Pic}^{2g-2}\widetilde{C}: h^0(\widetilde{C}, D) > 0\},$$
$$\Theta': = \{D \in \operatorname{Pic}^{g-1} C: h^0(C, D) > 0\}.$$

With these definitions, the inverse image of $\widetilde{\Theta}$ by the morphism $\pi^*: JC \to J\widetilde{C}$ is the divisor $\Theta_\beta + \Theta_{\beta^{-1}}$. The Prym variety (P, Ξ) of the double cover $\pi: \widetilde{C} \to C$ is defined by the reduced varieties $P: = P'_{(\kappa\otimes\beta)^{-1}}$ and $\Xi: = \Xi'_{(\kappa\otimes\beta)^{-1}}$ with

$$P': = \{E \in \operatorname{Pic}^{2g-2}\widetilde{C}: \operatorname{Nm}(E) \cong \omega_C, h^0(\widetilde{C}, E) \equiv 0 \bmod 2\},$$
$$\Xi': = \{E \in P': h^0(\widetilde{C}, E) > 0\},$$

where

$$\operatorname{Nm} : \operatorname{Pic}\widetilde{C} \longrightarrow \operatorname{Pic} C$$

is the Norm map (see [M2, p. 331–333 and p. 340–342]). As divisors we have

$$2\Xi = P \cdot \widetilde{\Theta}.$$

For any $E \in P'$, since $\operatorname{Nm}(E) \cong \omega_C$, we have $\omega_{\widetilde{C}} \otimes E^{-1} \cong \sigma^* E$. By the theorem of the square, the divisor $\Xi'_{E^{-1}} + \Xi'_{\sigma^* E^{-1}}$ is in the linear system $|2\Xi|$ and, by Wirtinger Duality (see [M2, p. 335–336]), such divisors span $|2\Xi|$. Furthermore, if $E \in \Xi'$, then $\Xi'_{E^{-1}} + \Xi'_{\sigma^* E^{-1}}$ is in $|2\Xi|_0$ and such divisors span $|2\Xi|_0$ (by Wirtinger duality, the span of such divisors is the span of $\phi(\Xi)$ where $\phi: P \to |2\Xi|^*$ is the natural morphism, this span is a hyperplane in $|2\Xi|^*$ which can therefore be identified with $|2\Xi|_0$ by Wirtinger duality, also see [W1, p. 18]).

We now explain how Proposition 7 follows from results of Welters and Debarre and how Proposition 9 follows from results of Debarre.

In this paragraph only, suppose that (P, Ξ) is a *general* Prym variety.

Then E is an element of the singular locus $\operatorname{Sing}(\Xi')$ of Ξ' if and only if $h^0(\widetilde{C}, E) \geq 4$ (see [W2, p. 168]). Therefore, for every $E \in \operatorname{Sing}(\Xi')$ and $(p, q) \in \widetilde{C}^2$, we have

$$h^0\big(\widetilde{C}, E \otimes \mathcal{O}_{\widetilde{C}}(p + q - \sigma p - \sigma q)\big) > 0.$$

Hence $\Sigma \subset \Xi'_{E^{-1}}$ and $\Xi'_{E^{-1}} + \Xi'_{\sigma^* E^{-1}}$ is in $|2\Xi|'_{00}$. Since Ξ is symmetric, the tangent cones at 0 to $\Xi'_{E^{-1}}$ and $\Xi'_{\sigma^* E^{-1}}$ are equal. It follows from these facts and the irreducibility of $\operatorname{Sing}(\Xi')$ for $p \geq 6$ (see [De, Thm

1.1, p. 114]) that V'_{00} is set-theoretically contained in $\bigcap_{E\in \mathrm{Sing}(\Xi')} \Xi'_{E^{-1}}$ and $V'_{\mathrm{inf},00}$ is set-theoretically contained in the intersection of the tangent cones at 0 to $\Xi'_{E^{-1}}$ for E a point of multiplicity 2 on Ξ'. This latter can be rephrased as: $V'_{\mathrm{inf},00}$ is set-theoretically contained in the intersection of the tangent cones to Ξ' at its points of multiplicity 2. It is proved in [W2, Thm 2.6, p. 169]) that, for $p \geq 16$, the intersection $\bigcap_{E\in \mathrm{Sing}(\Xi')} \Xi'_{E^{-1}}$ is the set-theoretical union of Σ and, possibly, some curves and points. This proves Proposition 7. By [De, Thm 1.1, p. 114], the tangent cones to Ξ' at its points of multiplicity 2 generate the space of quadrics containing χC for $p \geq 6$ and, for $p \geq 8$ (see [De, Cor. 2.3, p. 129], also see [L] and [LS]), the Prym-canonical curve χC is cut out by quadrics. This proves Proposition 9.

2. Pull-backs of divisors to $\widetilde{C}^2$

Let $\rho: \widetilde{C}^2 \to P \subset J\widetilde{C}$ be the morphism

$$\rho: (s,t) \longmapsto [s,t]: = \mathcal{O}_{\widetilde{C}}(s+t-\sigma s-\sigma t)$$

so that Σ is the image of $\widetilde{C}^2$ by ρ. The morphism ρ lifts to a morphism

$$\widetilde{\rho}: \widetilde{C}^2 \twoheadrightarrow \widetilde{\Sigma} \subset \widetilde{P}.$$

We have:

LEMMA 2.1. — *Let $E \in \Xi'$ be such that $h^0(\widetilde{C},E) = 2$ and let B be the base divisor of $|E|$. Then the inverse image of $\Xi'_{E^{-1}}$ in $\widetilde{C}^2$ is the divisor*

$$\rho^*\Xi'_{E^{-1}} = D_{\sigma^*E} + \Delta'$$

with

$$D_E = D_{E\otimes\mathcal{O}_{\widetilde{C}}(-B)} + p_1^*B + p_2^*B$$

where $p_i: \widetilde{C}^2 \to \widetilde{C}$ is the projection onto the i-th factor, the divisor $D_{E\otimes\mathcal{O}_{\widetilde{C}}(-B)}$ is reduced and equal to

$$D_{E\otimes\mathcal{O}_{\widetilde{C}}(-B)}: = \big\{(p,q): h^0(\widetilde{C},E\otimes\mathcal{O}_{\widetilde{C}}(-B-p-q)) > 0\big\}$$

and Δ' is the "pseudo-diagonal" of $\widetilde{C}^2$, i.e., the reduced curve

$$\Delta': = \big\{(p,\sigma p) \in \widetilde{C}^2\big\} = \rho^{-1}(0).$$

Furthermore, the divisor $\rho^*\Xi'_{E^{-1}}$ is in the linear system

$$\big|p_1^*\sigma^*E \otimes p_2^*\sigma^*E \otimes \mathcal{O}_{\widetilde{C}}(-\Delta+\Delta')\big|.$$

Proof. — We have the equality of sets

$$\rho^*\Xi'_{E^{-1}} = \{(p,q) : E\otimes[p,q]\in\Xi'\} = \{(p,q) : h^0(\widetilde{C}, E\otimes[p,q]) > 0\}.$$

This first implies that $\rho^*\Xi'_{E^{-1}}$ is a divisor: for general points p and q in $\widetilde{C}$, we have

$$h^0\big(\widetilde{C}, E\otimes\mathcal{O}_{\widetilde{C}}(p+q)\big) = 2 \quad\text{and}\quad h^0\big(\widetilde{C}, E\otimes\mathcal{O}_{\widetilde{C}}(-\sigma p-\sigma q)\big) = 0.$$

Therefore p and q are base points for

$$|E\otimes\mathcal{O}_{\widetilde{C}}(p+q)|$$

and

$$h^0(\widetilde{C}, E\otimes\mathcal{O}_{\widetilde{C}}(p+q-\sigma p-\sigma q)) = 0.$$

Secondly, if $[p,q] = \mathcal{O}_{\widetilde{C}}$, *i.e.*, $p=\sigma q$, or if

$$h^0\big(\widetilde{C}, E\otimes\mathcal{O}_{\widetilde{C}}(-B-\sigma p-\sigma q)\big) > 0,$$

then $[p,q]\in\rho^*\Xi'_{E^{-1}}$. So $\rho^*\Xi'_{E^{-1}} - D_{\sigma^*E\otimes\mathcal{O}_{\widetilde{C}}(-\sigma B)} - \Delta'$ is effective.

Now, if B is zero, an easy degree computation on the fibers of $\widetilde{C}^2$ over $\widetilde{C}$ by the two projections shows that $\rho^*\Xi'_{E^{-1}} = D_{\sigma^*E} + \Delta'$ as divisors.

Restricting to fibers of p_1 and p_2 and using the See-Saw Theorem, we see that $D_{\sigma^*E\otimes\mathcal{O}_{\widetilde{C}}(-\sigma B)}$ is in the linear system

$$\big|\,p_1^*(\sigma^*E\otimes\mathcal{O}_{\widetilde{C}}(-\sigma B))\otimes p_2^*(\sigma^*E\otimes\mathcal{O}_{\widetilde{C}}(-\sigma B))\otimes\mathcal{O}_{\widetilde{C}}(-\Delta)\big|$$

where Δ is the diagonal of $\widetilde{C}^2$.

Therefore $\rho^*\Xi'_{E^{-1}}$ is in the linear system

$$\big|\,p_1^*\sigma^*E\otimes p_2^*\sigma^*E\otimes\mathcal{O}_{\widetilde{C}}(-\Delta+\Delta')\big|$$

when B is zero, and, by continuity, also when B is non-zero.

So $\rho^*\Xi'_{E^{-1}} - D_{\sigma^*E\otimes\mathcal{O}_{\widetilde{C}}(-\sigma B)} - \Delta'$ is linearly equivalent to

$$p_1^*(\sigma B) + p_2^*(\sigma B)$$

and is effective. Since, by the Künneth isomorphism, the linear system $|p_1^*(\sigma B) + p_2^*(\sigma B)|$ has only one element, it follows that

$$\rho^*\Xi'_{E^{-1}} - D_{\sigma^*E\otimes\mathcal{O}_{\widetilde{C}}(-\sigma B)} - \Delta' = p_1^*(\sigma B) + p_2^*(\sigma B)$$

as divisors. $\square$

REMARK 2.2. — Suppose that E as above can be written as

$$E = \pi^* M \otimes \mathcal{O}_{\widetilde{C}}(B)$$

where M is an invertible sheaf on C with $h^0(C, M) = 2$. Then

$$D_{\sigma^* E} = \Delta' + D'$$

for some effective divisor D'. Therefore

$$\rho^* \Xi'_{E^{-1}} = D' + 2\Delta'$$

which agrees with the fact that in such a case $E \in \mathrm{Sing}(\Xi')$ so that $\Xi'_{E^{-1}}$ is singular at 0 (see [M2, p. 342–343]).

Let $\omega_{\widetilde{C}^2}$ be the canonical sheaf of $\widetilde{C}^2$. Then

$$\omega_{\widetilde{C}^2} \cong p_1^* \omega_{\widetilde{C}} \otimes p_2^* \omega_{\widetilde{C}}$$

and we have Künneth's isomorphism $H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2}) \cong H^0(\widetilde{C}, \omega_{\widetilde{C}})^{\otimes 2}$. Let

$$I_2(\widetilde{C}) \subset S^2 H^0(\widetilde{C}, \omega_{\widetilde{C}}) \subset H^0(\widetilde{C}, \omega_{\widetilde{C}})^{\otimes 2} = H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2})$$

be the vector space of quadratic forms vanishing on the canonical image $\kappa\widetilde{C}$ of $\widetilde{C}$. Fix an embedding $H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta)) \subset H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2})$ obtained by multiplication by a nonzero global section of $\mathcal{O}_{\widetilde{C}^2}(2\Delta)$ (note that $h^0(\widetilde{C}^2, \mathcal{O}_{\widetilde{C}^2}(2\Delta)) = 1$ because Δ has negative self-intersection, hence any two such embeddings differ by multiplication by a constant). Then it is easily seen that

$$I_2(\widetilde{C}) = S^2 H^0(\widetilde{C}, \omega_{\widetilde{C}}) \cap H^0\big(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta)\big) \subset H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2}).$$

Similarly fix embeddings

$$\begin{aligned} H^0\big(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta - 2\Delta')\big) &\subset H^0\big(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta)\big) \\ &\subset H^0\big(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta + 2\Delta')\big). \end{aligned}$$

For $E \in \Xi'$ such that $h^0(\widetilde{C}, E) = 2$, it is well-known (see, *e.g.*, [ACGH, p. 261]) that

$$q_E := \bigcup_{D \in |E|} \langle D \rangle = \bigcup_{D \in |\sigma^* E|} \langle D \rangle$$

is a quadric of rank ≤ 4 whose ruling(s) cut(s) the divisors of the moving parts of $|E|$ and $|\sigma^* E|$ on $\widetilde{C}$. We need the following:

LEMMA 2.3.

1) *We have*

$$\rho^*\mathcal{O}_P(2\Xi) \cong \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta + 2\Delta'),$$

$$\rho^*\Gamma_0 \subset I_2(\widetilde{C}) \subset H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta)) \subset H^0\big(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta + 2\Delta')\big),$$

$$\rho^*\Gamma_{00} \subset I_2(\widetilde{C}) \cap H^0\big(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta - 2\Delta')\big) \subset H^0\big(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta + 2\Delta')\big).$$

2) *For any* $f \in I_2(\widetilde{C}) \subset H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta + 2\Delta'))$, *let* $q(f)$ *be the quadric in* $|\omega_{\widetilde{C}}|^*$ *with equation* f. *Then, in* $\widetilde{C}^2$, *the zero locus of* $f \in H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta + 2\Delta'))$ *is*

$$Z(f) = Z_{q(f)} + 2\Delta'$$

where $Z_{q(f)}$ *is a divisor with support*

$$\big\{(p,q) \in \widetilde{C}^2 : \langle p + q\rangle \subset q(f)\big\}.$$

For any $s \in \Gamma_0$, *put*

$$q(s) = q(\rho^* s).$$

For a general $s \in \Gamma_0$, *the divisor* $Z_{q(s)}$ *is reduced. In particular, for* f *general, the divisor* $Z_{q(f)}$ *is reduced. If* $Z(s) = \Xi'_{E^{-1}} + \Xi'_{\sigma^* E^{-1}}$ *for some* $E \in \Xi'$ *such that* $h^0(\widetilde{C}, E) = 2$, *then* $q(s) = q_E := \bigcup_{D\in|E|} \langle D\rangle$.

3) *If* $s \in \Gamma_0 \setminus \Gamma_{00}$, *then*

$$q(s) \cap \big(\mathbb{P}T_0P = |\omega_C \otimes \alpha|^*\big) \subset \mathbb{P}T_0J\widetilde{C} = |\omega_{\widetilde{C}}|^*$$

is the projectivized tangent cone $\tau_{Z(s)}$ *to* $Z(s) \subset P$ *at* 0.

4) *For any* $s \in \Gamma_0$, *the multiplicity of* $\rho^* s$ *at the generic point of* Δ' *is even* ≥ 2 *and if* $\rho^* s$ *vanishes on* Δ' *with multiplicity* ≥ 4, *then either* $s \in \Gamma_{00}$ *or* $\tau_{Z(s)}$ *contains the Prym-canonical curve* χC.

Proof. — 1) Let E be an invertible sheaf of degree $2g-2$ on $\widetilde{C}$ such that $h^0(\widetilde{C}, E) = 2$ and $E \otimes \sigma^* E \cong \omega_{\widetilde{C}}$. Then, by Lemma 2.1, we have

$$\begin{aligned}\rho^*(\Xi'_{E^{-1}} &+ \Xi'_{\sigma^* E^{-1}}) \\ &= D_{\sigma^* E} + D_E + 2\Delta' \\ &\in \big| p_1^*\sigma^* E \otimes p_2^* \sigma^* E \otimes p_1^* E \otimes p_2^* E \otimes \mathcal{O}_{\widetilde{C}}(-2\Delta + 2\Delta')\big| \\ &= \big|\omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}}(-2\Delta + 2\Delta')\big|.\end{aligned}$$

This proves the first assertion.

Now, let s_1 and s_2 be two general sections of E. Then

$$s_1 \otimes s_2 - s_2 \otimes s_1 \in \Lambda^2 H^0(\widetilde{C}, E) \subset H^0(\widetilde{C}, E)^{\otimes 2}$$

and, as in the the proof of Lemma 2.1, it is easily seen that

$$Z(s_1 \otimes s_2 - s_2 \otimes s_1) = D_E + \Delta.$$

From the natural map

$$\psi_E : H^0(\widetilde{C}, E) \otimes H^0(\widetilde{C}, \sigma^* E) \longrightarrow H^0(\widetilde{C}, \omega_{\widetilde{C}})$$

we obtain the map

$$\begin{aligned} H^0(\widetilde{C}, E)^{\otimes 2} \otimes H^0(\widetilde{C}, \sigma^* E)^{\otimes 2} &\longrightarrow H^0(\widetilde{C}, \omega_{\widetilde{C}})^{\otimes 2} \cong H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2}), \\ t_1 \otimes t_2 \otimes \sigma^* u_1 \otimes \sigma^* u_2 &\longmapsto \psi_E(t_1 \otimes \sigma^* u_1) \otimes \psi_E(t_2 \otimes \sigma^* u_2), \end{aligned}$$

which induces the map

$$\phi_E : \Lambda^2 H^0(\widetilde{C}, E) \otimes \Lambda^2 H^0(\widetilde{C}, \sigma^* E) \longrightarrow S^2 H^0(\widetilde{C}, \omega_{\widetilde{C}}).$$

Put

$$t := (s_1 \otimes s_2 - s_2 \otimes s_1) \otimes (\sigma^* s_1 \otimes \sigma^* s_2 - \sigma^* s_2 \otimes \sigma^* s_1);$$

then $Z(\phi_E(t))$ is equal to $D_E + \Delta + D_{\sigma^* E} + \Delta$.

If $s \in \Gamma_0$ is such that $Z(s) = \Xi'_{E^{-1}} + \Xi'_{\sigma^* E^{-1}}$, then, by Lemma 2.1, we have

$$Z(\rho^* s) = D_E + D_{\sigma^* E} + 2\Delta'.$$

So $Z(\rho^* s) - 2\Delta' = Z(\phi_E(t)) - 2\Delta$ and the section $\rho^* s$ of

$$\rho^* \mathcal{O}_P(2\Xi) \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta') \cong \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta)$$

is a nonzero constant multiple of

$$\begin{aligned} \phi_E(t) \in S^2 H^0(\widetilde{C}, \omega_{\widetilde{C}}) &\cap H^0\big(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta)\big) \\ &= I_2(\widetilde{C}) \subset H^0\big(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta)\big). \end{aligned}$$

Since such s generate Γ_0, this proves that $\rho^* \Gamma_0 \subset I_2(\widetilde{C})$. The rest of part 1) easily follows now.

2) First think of f as an element of $S^2H^0(\widetilde{C},\omega_{\widetilde{C}}) \subset H^0(\widetilde{C}^2,\omega_{\widetilde{C}^2})$. We can write

$$f=\sum_{i=1}^{r} p_1^*\omega_i \otimes p_2^*\omega_i$$

for some $\omega_i \in H^0(\widetilde{C},\omega_{\widetilde{C}})$. Then, as sets,

$$Z(f)=\Big\{(p,q)\colon \sum_{i=1}^{r}\omega_i(p)\omega_i(q)=0\Big\}.$$

Since $f \in I_2(\widetilde{C})$, we have $\sum_{i=1}^{r}\omega_i(p)^2=0$ for every $p\in\widetilde{C}$. Therefore $Z(f)\supset\Delta$. Furthermore, for any two distinct points p and q of C, the equations

$$\sum_{i=1}^{r}\omega_i(p)^2=\sum_{i=1}^{r}\omega_i(p)\omega_i(q)=\sum_{i=1}^{r}\omega_i(q)^2=0$$

mean that the line $\langle p+q\rangle$ is in $q(f)$. Therefore

$$Z(f)=\big\{(p,q)\colon\langle p+q\rangle\subset q(f)\big\}\cup\Delta$$

as sets. Since $f\in S^2H^0(\widetilde{C},\omega_{\widetilde{C}})$, it vanishes with even multiplicity on Δ. So $Z(f)=Z_{q(f)}+2\Delta$ where $Z_{q(f)}$ has support $\{(p,q)\colon\langle p+q\rangle\subset q(f)\}$. Finally, if we think of f as a section of $\omega_{\widetilde{C}^2}\otimes\mathcal{O}_{\widetilde{C}^2}(-2\Delta+2\Delta')$, then $Z(f)=Z_{q(f)}+2\Delta'$.

Let s and E be as in 1). With the notation of 1), let X_1,X_2,X_3,X_4 be the images of, respectively,

$$s_1\otimes\sigma^*s_1,\, s_2\otimes\sigma^*s_1,\, s_1\otimes\sigma^*s_2,\, s_2\otimes\sigma^*s_2$$

by the map ψ_E. Then $\phi_E(t)=X_1X_4-X_2X_3$. By, *e.g.* [ACGH, p. 261], the polynomial $X_1X_4-X_2X_3$ is an equation for q_E. Therefore, since ρ^*s is a constant nonzero multiple of $\phi_E(t)$, we have $q(s)=q_E$. Hence

$$D_E+D_{\sigma^*E}=Z_{q_E}=Z_{q(s)}.$$

The subvariety of $\widetilde{C}^{(2g-2)}$ parametrizing divisors D such that $\mathcal{O}_{\widetilde{C}}(D)\in\Xi'$ maps dominantly to $|\omega_C|$ via $D\mapsto\pi_*D$. Therefore, for E general in Ξ', the linear systems $|E|$ and $|\sigma^*E|$ contain reduced divisors and D_E and D_{σ^*E} are reduced. Furthermore, the base loci B and σB have no points in common (because $\pi_*|E|$ contains reduced divisors). Therefore, if the moving parts of $|E|$ and $|\sigma^*E|$ are distinct, the divisors D_E and D_{σ^*E} have no common components and their sum is reduced. If this is not the case, then $E=\pi^*M\otimes\mathcal{O}_{\widetilde{C}}(B)$ for some effective line bundle M on C. Counting dimensions, we see that this does not happen for a general $E\in\Xi'$. Thus, if $Z(s)=\Xi'_{E^{-1}}+\Xi'_{\sigma^*E^{-1}}$ for $E\in\Xi'$ general, then $Z_{q(s)}$ is a reduced divisor. Hence $Z_{q(s)}$ is reduced for general $s\in\Gamma_0$.

3) When E is a smooth point of Ξ', the intersection $q_E \cap \mathbb{P}T_0P = 2\mathbb{P}T_E\Xi'$ is the projectivized tangent cone at 0 to $\Xi'_{E^{-1}} + \Xi'_{\sigma^* E^{-1}}$ (see [M2, p. 342–343]). Now 3) follows by linearity.

4) This immediately follows from the facts that

$$\rho^* s \in I_2(\widetilde{C}) \subset S^2 H^0(\widetilde{C}, \omega_{\widetilde{C}}) \subset H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2})$$

and that χC is the tangent cone at 0 to Σ. $\square$

3. More divisors in $|2\Xi|_{00}$

For any $M \in \operatorname{Pic}^{g-1} C$ we have:

PROPOSITION 3.1. — *The divisor $P \cdot \widetilde{\Theta}'_{\pi^* M^{-1}}$ is in the linear system $|2\Xi|$. It is in $|2\Xi|_0$ if $h^0(C, M)$ is positive.*

Proof. — We first prove that all the divisors $P \cdot \widetilde{\Theta}'_{\pi^* M^{-1}}$ are linearly equivalent as M varies in $\operatorname{Pic}^{g-1} C$. Let $\psi: JC \to \operatorname{Pic}^0 P$ be the morphism of abelian varieties which sends $M \otimes \kappa^{-1} \otimes \beta^{-1}$ to

$$\mathcal{O}_P(P \cdot \widetilde{\Theta}'_{\pi^* M^{-1}} - P \cdot \widetilde{\Theta}) \in \operatorname{Pic}^0 P.$$

Then the map ψ is the dual of the zero map $P \hookrightarrow J\widetilde{C} \xrightarrow{\mathrm{Nm}} JC$. Hence the image of ψ is $\mathcal{O}_P$. Since $P \cdot \widetilde{\Theta} = 2\Xi$, all the divisors $P \cdot \widetilde{\Theta}'_{\pi^* M^{-1}}$ are linearly equivalent to 2Ξ. The second assertion is now immediate. $\square$

PROPOSITION 3.2. — *The divisor $P \cdot \widetilde{\Theta}'_{\pi^* M^{-1}}$ is an element of $|2\Xi|_{00}$ if $h^0(C, M) \geq 2$.*

Proof. — If $h^0(\widetilde{C}, \pi^* M) > 2$, then $\widetilde{\Theta}'_{\pi^* M^{-1}}$ has multiplicity at least three at the origin whence so has its restriction $P \cdot \widetilde{\Theta}'_{\pi^* M^{-1}}$. Since this multiplicity is even, it is at least 4.

Suppose therefore that $h^0(\widetilde{C}, \pi^* M) = 2$. By, *e.g.* [ACGH, p. 261], the tangent cone to $\widetilde{\Theta}'_{\pi^* M^{-1}}$ at 0 is the quadric

$$q_{\pi^* M} := \bigcup_{\delta \in |\pi^* M|} \langle \delta \rangle$$

in $|\omega_{\widetilde{C}}|^* = \mathbb{P}T_0 J\widetilde{C}$. Let π also denote the projection

$$\mathbb{P}T_0 J\widetilde{C} = |\omega_{\widetilde{C}}|^* \longrightarrow |\omega_C|^* = \mathbb{P}T_0 JC$$

with center $|\omega_C \otimes \alpha|^* = \mathbb{P}T_0P$. Since

$$q_{\pi^*M} = \pi^*(q_M) = \pi^*\Big(\bigcup_{\delta\in|M|}\langle\delta\rangle\Big),$$

q_{π^*M} contains $\mathbb{P}T_0P \subset \mathbb{P}T_0J\widetilde{C}$ and the multiplicity of $P\cdot\widetilde{\Theta}'_{\pi^*M^{-1}}$ at 0 is at least 3. Since this multiplicity is even, we have $P\cdot\widetilde{\Theta}'_{\pi^*M^{-1}} \in |2\Xi|_{00}$. □

4. The base locus of $L_{|\widetilde{\Sigma}}$

In this section we prove Proposition 5 and Theorem 3. We will use the divisors of $|2\Xi|_{00}$ that we constructed in Section 3. We have:

PROPOSITION 4.1. — *If $M \in \operatorname{Pic}^{g-1} C$ is such that*

$$h^0(C,M) = h^0(\widetilde{C},\pi^*M) = 2,$$

then

$$\rho^*(P\cdot\widetilde{\Theta}'_{\pi^*M^{-1}}) = D_{\pi^*M} + D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}} + 2\Delta'$$

where D_{π^*M} is defined as in Lemma 2.1. Furthermore, the divisors $D_{\pi^*M} - \Delta'$ and $D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}} - \Delta'$ are effective with respective supports

$$\big\{(p,q) : h^0(C,M\otimes\mathcal{O}_C(-\pi p - \pi q)) > 0\big\}$$

and

$$\big\{(p,q) : h^0(C,\omega_C\otimes M^{-1}\otimes\mathcal{O}_C(-\pi p - \pi q)) > 0\big\}$$

Proof. — First note that $P\cdot\widetilde{\Theta}'_{\pi^*M^{-1}}$ does not contain Σ. Indeed, for general points p and q in $\widetilde{C}$, we have

$$h^0\big(\widetilde{C},\pi^*M\otimes\mathcal{O}_{\widetilde{C}}(p+q)\big) = 2 \quad\text{and}\quad h^0\big(\widetilde{C},\pi^*M\otimes\mathcal{O}_{\widetilde{C}}(-\sigma p - \sigma q)\big) = 0.$$

Therefore p and q are base points for

$$\big|\pi^*M\otimes\mathcal{O}_{\widetilde{C}}(p+q)\big|$$

and

$$h^0\big(\widetilde{C},\pi^*M\otimes\mathcal{O}_{\widetilde{C}}(p+q-\sigma p-\sigma q)\big) = 0.$$

Restricting to fibers of p_1 and p_2 and using the See-Saw Theorem, we see that

$$D_{\pi^*M} + D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}} + 2\Delta' \in \big|\omega_{\widetilde{C}^2}(-2\Delta+2\Delta')\big|.$$

Hence, by Lemma 2.3 and Proposition 3.1, the divisor

$$D_{\pi^* M} + D_{\omega_{\widetilde{C}} \otimes \pi^* M^{-1}} + 2\Delta'$$

is linearly equivalent to $\rho^*(P \cdot \widetilde{\Theta}'_{\pi^* M^{-1}})$. Let B and B' be the respective base loci of $|M|$ and $|\omega_C \otimes M^{-1}|$. By definition the support of $\rho^*(P \cdot \widetilde{\Theta}'_{\pi^* M^{-1}})$ is the set

$$\big\{(p,q) : h^0(\widetilde{C}, \pi^* M \otimes \mathcal{O}_{\widetilde{C}}(p + q - \sigma p - \sigma q)) > 0\big\}$$

which, by Riemann-Roch and Serre Duality is equal to the set

$$\big\{(p,q) : h^0(\widetilde{C}, \omega_{\widetilde{C}} \otimes \pi^* M^{-1} \otimes \mathcal{O}_{\widetilde{C}}(\sigma p + \sigma q - p - q)) > 0\big\}.$$

Therefore the support of $\rho^*(P \cdot \widetilde{\Theta}'_{\pi^* M^{-1}})$ contains the sets

$$\big\{(p,q) : h^0(C, \pi^* M \otimes \mathcal{O}_{\widetilde{C}}(-\pi^* B - p - q)) > 0\big\} = D_{\pi^* M \otimes \mathcal{O}_{\widetilde{C}}(-\pi^* B)},$$

$$\big\{(p,q) : h^0(C, \omega_{\widetilde{C}} \otimes \pi^* M^{-1} \otimes \mathcal{O}_{\widetilde{C}}(-\pi^* B' - \pi p - \pi q)) > 0\big\} = D_{\omega_{\widetilde{C}} \otimes \pi^* M^{-1} \otimes \mathcal{O}_{\widetilde{C}}(-\pi^* B')}$$

and the support of $p_1^* \pi^* B + p_2^* \pi^* B + p_1^* \pi^* B' + p_2^* \pi^* B'$. Therefore, when B and B' are reduced, we have

$$\rho^*(P \cdot \widetilde{\Theta}'_{\pi^* M^{-1}}) = D_{\pi^* M \otimes \mathcal{O}_{\widetilde{C}}(-\pi^* B)} + D_{\omega_{\widetilde{C}} \otimes \pi^* M^{-1} \otimes \mathcal{O}_{\widetilde{C}}(-\pi^* B')} + 2\Delta' + p_1^* \pi^* B + p_2^* \pi^* B + p_1^* \pi^* B' + p_2^* \pi^* B'.$$

When B or B' is not reduced, choose a (flat) one-parameter family of divisors $\{B_t + B'_t\}_{t \in T}$ whose general member is reduced and which has a special member $B_0 + B'_0$, $0 \in T$ with $B_0 = B$ and $B'_0 = B'$. Then the two families of divisors $\rho^*(P \cdot \widetilde{\Theta}'_{\pi^* M^{-1} \otimes \mathcal{O}_{\widetilde{C}}(\pi^* B - \pi^* B_t)})$ and

$$D_{\pi^* M \otimes \mathcal{O}_{\widetilde{C}}(-\pi^* B)} + D_{\omega_{\widetilde{C}} \otimes \pi^* M^{-1} \otimes \mathcal{O}_{\widetilde{C}}(-\pi^* B')} + 2\Delta' + p_1^* \pi^* B_t + p_2^* \pi^* B_t + p_1^* \pi^* B'_t + p_2^* \pi^* B'_t$$

define two divisors on $\widetilde{C}^2 \times T$ which are equal because they do not contain $\widetilde{C}^2 \times \{0\}$ and are equal outside it.

The other assertions of the proposition are immediate. $\square$

PROPOSITION 4.2. — *Let $M \in \operatorname{Pic}^{g-1} C$ be such that*

$$h^0(M) = h^0(\pi^* M) = 2.$$

Let $s \in \Gamma_{00}$ be such that $Z(s) = P \cdot \widetilde{\Theta}'_{\pi^* M^{-1}}$. Then, with the notation of Lemma 2.3 we have

$$q(s) = \pi^* q_M.$$

Proof. — We have, by Proposition 4.1, that

$$\rho^*(P \cdot \widetilde{\Theta}'_{\pi^* M^{-1}}) = D_{\pi^* M} + D_{\omega_{\widetilde{C}} \otimes \pi^* M^{-1}} + 2\Delta'.$$

Noting that

$$\begin{aligned} D_{\pi^* M} \cup D_{\omega_{\widetilde{C}} \otimes \pi^* M^{-1}} &= \{(u,v) \colon \langle \pi u + \pi v \rangle \subset q_M\} \cup \Delta' \\ &= \{(u,v) \colon \langle u + v \rangle \subset \pi^* q_M\} \end{aligned}$$

as sets, it follows from Lemma 2.3 that $\pi^* q_M = q(s)$. $\square$

Let

$$\tau_2 \colon \Gamma_0 \longrightarrow H^0(\mathbb{P}T_0 P, \mathcal{O}_{\mathbb{P}T_0 P}(2))$$

be the map which to $s \in \Gamma_0$ associates the quadric term of its Taylor expansion at 0. Then τ_2 is onto because, since (P, Ξ) is indecomposable, every quadric of rank 1 can be obtained as the tangent cone at the origin to a divisor $\Xi_\gamma + \Xi_{\gamma^{-1}}$ for some $\gamma \in P$. We have:

LEMMA 4.3. — *For $s \in \Gamma_0$,*

$$s \in \Gamma_{00} \iff q(s) \supset \mathbb{P}T_0 P.$$

Proof. — By Lemma 2.3 part 3) and with the notation there, if $s \in \Gamma_0 \setminus \Gamma_{00}$, then $\tau_{Z(s)} = q(s) \cap \mathbb{P}T_0 P$. So the projectivizations of the two maps τ_2 and $s \mapsto (\rho^* s)_{|\mathbb{P}T_0 P} \in I_2(\widetilde{C})_{|\mathbb{P}T_0 P}$ are equal. Hence, there exists $\lambda \in \mathbb{C}^*$ such that, for every $s \in \Gamma_0$, we have $\lambda \tau_2(s) = (\rho^* s)_{|\mathbb{P}T_0 P}$. So

$$s \in \Gamma_{00} \iff \tau_2(s) = 0 \iff (\rho^* s)_{|\mathbb{P}T_0 P} = 0 \iff q(s) \supset \mathbb{P}T_0 P. \quad \square$$

Let $I_2(\widetilde{C}, \alpha)$ be the subvector space of $I_2(\widetilde{C})$ consisting of elements which vanish on $\mathbb{P}T_0 P$. By the above lemma and because all elements of Γ_0 are even, the map ρ^* sends Γ_{00} into the subspace $I_2(\widetilde{C}, \alpha)^+$ of σ-invariant elements of $I_2(\widetilde{C}, \alpha)$. We have:

LEMMA 4.4. — *The subspace* $I_2(\widetilde{C},\alpha)^+$ *is equal to* $I_2(C) \overset{\pi^*}{\subset} I_2(\widetilde{C})$.

Proof. — The σ-invariant and σ-anti-invariant parts of $H^0(\widetilde{C},\omega_{\widetilde{C}})$ are, respectively, $H^0(C,\omega_C)$ and $H^0(C,\omega_C\otimes\alpha)$. Therefore, in the decomposition

$$\begin{aligned}&S^2H^0(\widetilde{C},\omega_{\widetilde{C}})\\&\quad= S^2H^0(C,\omega_C)\oplus H^0(C,\omega_C)\otimes H^0(C,\omega_C\otimes\alpha)\oplus S^2H^0(C,\omega_C\otimes\alpha),\end{aligned}$$

the space $S^2H^0(C,\omega_C)\oplus S^2H^0(C,\omega_C\otimes\alpha)$ is the σ-invariant part of $S^2H^0(\widetilde{C},\omega_{\widetilde{C}})$. So $S^2H^0(C,\omega_C)$ is the subspace of σ-invariant elements of $S^2H^0(\widetilde{C},\omega_{\widetilde{C}})$ which vanish on $\mathbb{P}T_0P$. Therefore $I_2(\widetilde{C},\alpha)^+$ is contained in $S^2H^0(C,\omega_C)$ and $I_2(\widetilde{C},\alpha)^+$ is the subspace of elements of $S^2H^0(C,\omega_C)$ which vanish on $\kappa\widetilde{C}$. This is precisely $I_2(C)$. ▯

Let W^1_{g-1} be the subvariety of $\operatorname{Pic}^{g-1}C$ parametrizing invertible sheaves M with $h^0(C,M)\geq 2$. We have:

COROLLARY 4.5. — *The pull-back*

$$\rho^*:\Gamma_{00}\longrightarrow I_2(C)\big(\overset{\pi^*}{\subset} I_2(\widetilde{C})\big)$$

is onto.

Proof. — Since all quadrics of rank four containing κC are of the form q_M for some $M\in W^1_{g-1}$ and a general such M has the properties required in Proposition 4.2, this follows from Proposition 4.2 and the fact that quadrics of rank four generate $|I_2(C)|$ (see [G] and [SV]). ▯

Note that, since all elements of Γ_0 are even, the map ρ^* sends Γ_0 into the subspace $I_2(\widetilde{C})^+$ of σ-invariant elements of $I_2(\widetilde{C})$. We have:

COROLLARY 4.6. — *The pull-back*

$$\rho^*:\Gamma_0\longrightarrow I_2(\widetilde{C})^+\big(\subset I_2(\widetilde{C})\big)$$

is surjective.

Proof. — Take the σ-invariants of the exact sequence

$$0\to I_2(\widetilde{C})\longrightarrow S^2H^0(\widetilde{C},\omega_{\widetilde{C}})\longrightarrow H^0\big(\widetilde{C},\omega_{\widetilde{C}}^{\otimes 2}\big)\to 0$$

to obtain the exact sequence

$$0 \to I_2(\widetilde{C})^+ \longrightarrow S^2H^0(C,\omega_C) \oplus S^2H^0(C,\omega_C \otimes \alpha) \longrightarrow H^0(C,\omega_C^{\otimes 2}) \to 0.$$

Taking the quotient of this sequence by $I_2(C)$, we obtain

$$0 \to \frac{I_2(\widetilde{C})^+}{I_2(C)} \longrightarrow H^0(C,\omega_C^{\otimes 2}) \oplus S^2H^0(C,\omega_C \otimes \alpha) \longrightarrow H^0(C,\omega_C^{\otimes 2}) \to 0$$

or

$$\frac{I_2(\widetilde{C})^+}{I_2(C)} \xrightarrow{\cong} S^2H^0(C,\omega_C \otimes \alpha)$$

where the isomorphism is obtained from the restriction to $\mathbb{P}T_0P$. By Corollary 4.5, we have the equality $\rho^*\Gamma_{00} = I_2(C)$ and hence the embedding

$$\frac{\rho^*\Gamma_0}{I_2(C)} \hookrightarrow \frac{I_2(\widetilde{C})^+}{I_2(C)} = S^2H^0(C,\omega_C \otimes \alpha)$$

which is equal to the map $s \mapsto \rho^*(s)_{|\mathbb{P}T_0P}$. The projectivization of this map is equal to the projectivization of τ_2 (see Lemmas 2.3 and 4.3). Since τ_2 is surjective, we have $\dfrac{\rho^*\Gamma_0}{I_2(C)} = \dfrac{I_2(\widetilde{C})^+}{I_2(C)}$ and $\rho^*\Gamma_0 = I_2(\widetilde{C})^+$. $\square$

Remark 4.7 (C. Pauly). — Counting dimensions, one easily deduces from the above two corollaries that Γ'_{00} is in fact the subspace of elements of Γ (and not just Γ_{00}) vanishing on Σ.

The following implies Theorem 3.

Proposition 4.8. — *The inverse image by $\widetilde{\rho}$ of the support of the base locus of $L_{|\widetilde{\Sigma}}$ is the set of elements (p,q) of $\widetilde{C}^2$ such that $\langle \pi p + \pi q\rangle$ is contained in the intersection of the quadrics containing the canonical curve κC. In particular, if C is not trigonal, then the base locus of L does not intersect $\widetilde{\Sigma}$.*

Proof. — By Proposition 3.2 and Corollary 4.5, the base locus of $L_{|\widetilde{\Sigma}}$ is supported on

$$\widetilde{\Sigma} \cap \Big(\bigcap_{M \in W^1_{g-1}} \epsilon_*^{-1}(P \cdot \widetilde{\Theta}'_{\pi^*M^{-1}})\Big)$$

where

$$\epsilon_*^{-1}(P \cdot \widetilde{\Theta}'_{\pi^*M^{-1}}) = \epsilon^*(P \cdot \widetilde{\Theta}'_{\pi^*M^{-1}}) - 4\mathcal{E}.$$

We have

$$\widetilde{\rho}^*(\epsilon_*^{-1}(P\cdot\widetilde{\Theta}'_{\pi^*M^{-1}}))=\rho^*(P\cdot\widetilde{\Theta}'_{\pi^*M^{-1}})-4\Delta'.$$

Since $\langle \pi p+\pi q\rangle\subset q_M$ is equivalent to $h^0(C,M\otimes\mathcal{O}_C(-\pi p-\pi q))>0$ or $h^0(C,\omega_C\otimes M^{-1}\otimes\mathcal{O}_C(-\pi p-\pi q))>0$, it follows from Proposition 4.1 that the inverse image

$$\widetilde{\rho}^*\Big(\bigcap_{M\in W^1_{g-1}}\epsilon_*^{-1}(P.\widetilde{\Theta}'_{\pi^*M^{-1}})\Big)$$

is supported on the set of elements (p,q) of $\widetilde{C}^2$ such that $\langle\pi p+\pi q\rangle$ is contained in q_M for all $M\in W^1_{g-1}$. Since the quadrics of the form q_M generate $|I_2(C)|$ (see [G] and [SV]) and the base locus of $|I_2(C)|$ in the canonical space $|\omega_C|^*$ does not contain any secants to κC for C non-trigonal (see [ACGH, p. 124]), the proposition follows. ▯

Corollary 4.9. — *The dimension of Γ'_{00} is equal to 2^p-2-p^2+p. The codimension of $\mathcal{Q}'_{00}$ in $\mathcal{Q}_{00}$ is at least $g-3$.*

Proof. — By Lemma 4.4, we have

$$\Gamma'_{00}=\operatorname{Ker}\big(\rho^*:\Gamma_{00}\to I_2(C)\overset{\pi^*}{\subset} I_2(\widetilde{C})\big).$$

Since ρ^* maps Γ_{00} onto $I_2(C)$ by Corollary 4.5, the dimension of Γ'_{00} is equal to

$$\begin{aligned}\dim(\Gamma_{00})-\dim(I_2(C))&=2^p-1-\tfrac12 p(p+1)-\tfrac12(g-2)(g-3)\\&=2^p-1-\tfrac12 p(p+1)-\tfrac12(p-1)(p-2)\\&=2^p-2-p^2+p.\end{aligned}$$

The linear system $\widetilde{\rho}^*L$ contains the divisors $\widetilde{\rho}^*(P\cdot\widetilde{\Theta}'_{\pi^*M^{-1}})-4\Delta'$ for $M\in W^1_{g-1}$. From now on we suppose that

$$h^0(C,M)=h^0(\widetilde{C},\pi^*M)=2.$$

By Proposition 4.1, we have

$$\widetilde{\rho}^*(P\cdot\widetilde{\Theta}'_{\pi^*M^{-1}})-4\Delta'=D_{\pi^*M}-\Delta'+D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}}-\Delta'.$$

Therefore the restriction of $\widetilde{\rho}^*L$ to $\Delta'\cong\widetilde{C}$ contains the divisors

$$(D_{\pi^*M}-\Delta')_{|\Delta'}+(D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}}-\Delta')_{|\Delta'}.$$

We will prove that the map $M \mapsto (\widetilde{\rho}^*(P\cdot\widetilde{\Theta}'_{\pi^*M^{-1}})-4\Delta')_{|\Delta'}$ is generically finite. This will imply that the set of such divisors on $\Delta' \cong \widetilde{C}$ has dimension $g-4$ and the codimension of $\mathcal{Q}'_{00}$ in $\mathcal{Q}_{00}$ is at least $g-4+1 = g-3$. We let R and R' be the ramification divisors of the natural maps $C \to |M|^* \cong \mathbb{P}^1$ and $C \to |\omega_C \otimes M^{-1}|^* \cong \mathbb{P}^1$ respectively

First suppose C non-trigonal. If C is bielliptic and $g \geq 6$, we choose M in the component of W^1_{g-1} whose general elements are base-point-free. For M general, the linear systems $|M|$ and $|\omega_C \otimes M^{-1}|$ have no base points and

$$(D_{\pi^*M} - \Delta')_{|\Delta'} = \pi^*R, \quad (D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}} - \Delta')_{|\Delta'} = \pi^*R'.$$

It follows that the map $M \mapsto (\widetilde{\rho}^*(P\cdot\widetilde{\Theta}'_{\pi^*M^{-1}})-4\Delta')_{|\Delta'}$ is generically finite.

Now suppose C trigonal. Since $g \geq 5$, the curve C has a unique linear system of degree 3 and dimension 1 and we denote the associated invertible sheaf of degree 3 by M_0. Choose a general effective divisor N of degree $g-4$ on C and put $M = M_0 \otimes \mathcal{O}_C(N)$. We have

$$\begin{aligned}\widetilde{\rho}^*(P\cdot\widetilde{\Theta}'_{\pi^*M^{-1}})-4\Delta' = D_{\pi^*M_0} &- \Delta' + p_1^*(\pi^*N) \\ &+ p_2^*(\pi^*N) + D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}} - \Delta',\end{aligned}$$

$$\begin{aligned}(\widetilde{\rho}^*(P\cdot\widetilde{\Theta}'_{\pi^*M^{-1}})-4\Delta')_{|\Delta'} = (D_{\pi^*M_0} &- \Delta')_{|\Delta'} \\ &+ 2\pi^*N + (D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}} - \Delta')_{|\Delta'}\end{aligned}$$

(where we identify Δ' with $\widetilde{C}$). Since N is general, the linear system $|\omega_C \otimes M^{-1}|$ has no base points and

$$(D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}} - \Delta')_{|\Delta'} = \pi^*R'.$$

It follows once more that the map $M \mapsto (\widetilde{\rho}^*(P\cdot\widetilde{\Theta}'_{\pi^*M^{-1}})-4\Delta')_{|\Delta'}$ is generically finite. □

BIBLIOGRAPHIE

[ACGH] ARBARELLO (E.), CORNALBA (M.), GRIFFITHS (P.A.), HARRIS (J.). — *Geometry of algebraic curves*, vol. 1. — Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1985.

[BD] BEAUVILLE (A.), DEBARRE (O.). — *Sur les fonctions thêta du second ordre – Arithmetic of Complex Manifolds.* — Lecture Notes in Math., vol. **1399**, Springer-Verlag, Berlin-New York, 1989, p. 27–39.

[BDDG] BEAUVILLE (A.), DEBARRE (O.), DONAGI (R.), VAN DER GEER (G.). — *Sur les fonctions thêta d'ordre deux et les singularités du diviseur thêta*, C.R. Acad. Sc. Paris, t. **307**, 1988, p. 481–484.

[De] DEBARRE (O.). — *Sur le problème de Torelli pour les variétés de Prym*, Amer. J. Math., t. **111**, 1989, p. 111–134.

[Do1] DONAGI (R.). — *Theory of Moduli.* — Lecture Notes in Math., vol. **1337**, Springer-Verlag, 1988, p. 84–137.

[Do2] DONAGI (R.). — *The fibers of the Prym map – Curves, Jacobians and Abelian Varieties* (Amherst, MA 1990). — Contemp. Math., vol. **136**, Amer. Math. Soc., 1992, p. 55–125.

[DS] DONAGI (R.), SMITH (R.). — *The structure of the Prym map*, Acta Math., t. **146**, 1981, p. 25–102.

[F] FAY (J.D.). — *On the even-order vanishing of Jacobian theta functions*, Duke Math. J., t. **51**, 1984, n° 1, p. 109–132.

[FS] FRIEDMAN (R.), SMITH (R.). — *The generic Torelli theorem for the Prym map*, Invent. Math., t. **67**, 1982, n° 3, p. 473–490.

[GG] VAN GEEMEN (B.), VAN DER GEER (G.). — *Kummer varieties and the moduli spaces of abelian varieties*, Amer. J. Math., t. **108**, 1986, p. 615–642.

[G] GREEN (M.L.). — *Quadrics of rank four in the ideal of a canonical curve*, Invent. Math., t. **75**, 1984, n° 1, p. 85–104.

[GH] GRIFFITHS (P.A.), HARRIS (J.). — *Principles of algebraic geometry.* — Wiley, New York, 1978.

[Ig] IGUSA (J.I.). — *Theta functions.* — Springer Verlag, Berlin, New York, Heidelberg, 1972.

[Iz1] IZADI (E.). — *Fonctions thêta du second ordre sur la jacobienne d'une courbe lisse*, Math. Ann., t. **289**, 1991, n° 2, p. 189–202.

[Iz2] IZADI (E.). — *The geometric structure of $\mathcal{A}_4$, the structure of the Prym map, double solids and Γ_{00}-divisors*, J. reine angew. Math., t. **462**, 1995, p. 93–158.

[LS] LANGE (H.), SERNESI (E.). — *Quadrics containing a Prym-canonical curve*, J. Alg. Geom., t. **5**, 1996, n° 2, p. 387–399.

[L] LAZARSFELD (R.). — *Intersection des quadriques contenant une courbe lisse* (rédigé par O. Debarre). — Preprint EUROPROJ.

[M1] MUMFORD (D.). — *Theta characteristics of an algebraic curve*, Ann. Sci. École Norm. Sup., t. **4**, 1971, n° 4, p. 181–192.

[M2] MUMFORD (D.). — *Prym varieties*, I. — Contributions to Analysis (L.V. Ahlfors, I. Kra, B. Maskit, L. Niremberg, eds.), Academic Press, 1974, p. 325–350.

[SV] SMITH (R.), VARLEY (R.). — *Deformations of theta divisors and the rank 4 quadrics problem*, Compositio Math., t. **76**, 1990, n° 3, p. 367–398.

[W1] WELTERS (G.E.). — *The surface $C - C$ on jacobi varieties and second order theta functions*, Acta Mathematica, t. **157**, 1986, p. 1–22.

[W2] WELTERS (G.E.). — *Recovering the curve data from a general prym variety*, Amer. J. Math., t. **109**, 1987, n° 1, p. 165–182.