ON THE MODULI SPACE OF FOUR-DIMENSIONAL PRINCIPALLY POLARIZED ABELIAN VARIETIES

by

Elham Izadi

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ABSTRACT

Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties (*ppav*) of dimension g over the field $\mathbb{C}$ of the complex numbers. Our aim is to study the structure of $\mathcal{A}_4$ and some of its relations with $\mathcal{A}_5$.

Let $\mathcal{J}_{hyp}$ be the locus of jacobians of hyperelliptic curves in $\mathcal{A}_4$ and let $\overline{\mathcal{J}}_{hyp}$ be its closure in $\mathcal{A}_4$. Our main tool will be a cubic threefold T with an "even" double cover $l : \tilde{F} = P^{-1}(A) \to F$ of the Fano variety of lines F of T which we will associate to each element A of $\mathcal{A}_4 \setminus (\overline{\mathcal{J}}_{hyp} \cup \mathcal{A}_{n11})$ (see the text for the definition of $\mathcal{A}_{n11}$). The pair (T,μ) (where μ is the point of order 2 associated to l) was first introduced by Donagi for generic ppav's in an abstract way. He used points of order 2 on plane quintics.

Using a construction of Clemens for double solids we first define (T,l) as a hypersurface in $|2\Theta|_{00}$ for a generic ppav A. Here Θ is a symmetric theta-divisor on A and $|2\Theta|_{00}$ is the sublinear system of $|2\Theta|$ consisting of those divisors which have a point of multiplicity greater than or equal to 4 at 0. Then we extend the definition of (T,l) to $\mathcal{A}_4 \setminus (\overline{\mathcal{J}}_{hyp} \cup \mathcal{A}_{n11})$.

Let $h : A \to (|2\Theta|_{00})^*$ be the natural map. Let $\tilde{A}$ be the blow up of A at 0 and let $\tilde{h}$ be the lift of h to $\tilde{A}$. From theorem 4 below it follows that $\tilde{h}$ is a morphism everywhere on $\mathcal{A}_4 \setminus (\overline{\mathcal{J}}_{hyp} \cup \mathcal{A}_{n11})$. Let E be the exceptional divisor in $\tilde{A}$. For a generic ppav we show :

Theorem 1 : *The branch locus of* $\tilde{h}$ *is the union of* $\tilde{h}(E)$ *and the dual variety* T^* *of* T.

There is a three-dimensional family of double solids (with six ordinary double points in general position) with intermediate jacobian a given ppav A. The first use of the cubic threefold will be

Theorem 2 : *One can recover all the double solids with intermediate jacobian* A *from the data of* A .

Let $\tau|2\Theta|_{00}$ be the linear system of *quartic* tangent cones to elements of $|2\Theta|_{00}$. In relation with the Schottky problem of characterizing jacobians of curves in $\mathcal{A}_4$ we prove :

Theorem 3 : *Let* $A \in \mathcal{A}_4 \setminus (\overline{\mathcal{J}}_4 \cup \mathcal{A}_{n11})$ *, then*

(i) *The only base point of* $|2\Theta|_{00}$ *is* 0 *with multiplicity* 4^4 .

(ii) *The base locus of* $\tau|2\Theta|_{00}$ *is empty. In particular,* $\tau|2\Theta|_{00}$ *has always dimension* ≥ 3 .

We also determine the loci on which T is singular. Let θ_{null} be the locus of ppav's with a vanishing theta-null.

Theorem 4 : Let $A \in \mathcal{A}_4 \setminus (\overline{\mathcal{J}}_{hyp} \cup \mathcal{A}_{n11})$. *The cubic threefold* T *associated to* A *is singular if and only if* $A \in \theta_{null} \cup \mathcal{J}_4$.

From this we deduce

Corollary : *There is a finite rational map of degree* $2^4(2^3+1) - 1 + 2^4(2^3+1) - 1$

$$\theta_{null} \quad \to \quad \mathcal{J}_4$$

Using the cubic threefold we then complete results of Clemens and Donagi by determining the double solids above a generic element of $\theta_{null} \cup \mathcal{J}_4$.

At the end, we gather a few more results that could be useful for further developments.

To my family

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1. INTRODUCTION

Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties (*ppav*) of dimension g over the field $\mathbb{C}$ of the complex numbers. Our aim is to study the structure of $\mathcal{A}_4$ and some of its relations with $\mathcal{A}_5$. We summarize some of our results in a diagram, which we now describe.

For a family of ppav's parametrized by a variety S we denote by S^0 the open subset of S parametrizing "automorphism free" abelian varieties, i.e., those whose only automorphisms are translations and $\pm$identity.

Let $\mathcal{K}$ be the universal Kummer variety over $\mathcal{A}_4^0$. For an element A of $\mathcal{A}_4$ with symmetric theta divisor Θ we denote by Γ the vector space $H^0(A, \mathcal{O}(2\Theta))$, by Γ_{00} the space of sections of Γ vanishing at 0 with multiplicity greater than or equal to 4 and by $|2\Theta|_{00}$ the projectivisation of Γ_{00}. For an indecomposable abelian variety, i.e., one that is not the product of two ppav's of lower dimensions, the dimension of Γ_{00} is 5 (see [I] p. 188). An automorphism free ppav is a fortiori indecomposable.

Let $\mathbb{P}$ be the bundle over $\mathcal{A}_4^0$ with fiber $|2\Theta|_{00}$ at A ; likewise $\mathbb{P}^*$ is the bundle with fibers $(|2\Theta|_{00})^*$. So we have a commutative diagram of canonical maps :

$$\begin{array}{ccc} \mathcal{K} & \rightarrow & \mathbb{P}^* \\ \downarrow & \swarrow & \\ \mathcal{A}_4^0 & & \end{array}$$

Let $\mathcal{P}_{g+1}$ be the moduli space of curves X of genus $g+1$ with a distinguished point η of order 2 in the jacobian JX of X. For (X, η) in $\mathcal{P}_{g+1}$, if $\tilde{X}$ is the double cover of X determined by η, then the covering involution $\sigma : \tilde{X} \rightarrow \tilde{X}$ induces an involution (still denoted by σ) on $J\tilde{X}$. The Prym variety $P(X,\eta)$ or $P(\tilde{X},X)$ is the image of $(\sigma - id) : J\tilde{X} \to J\tilde{X}$. The Prym variety is a ppav of dimension g . The map which to each (X,η) associates its Prym variety is called the Prym map and denoted by P . If for an abelian variety B we denote by B_2 its group of points of order 2 , we have an exact sequence ([M3]) :

$$0 \longrightarrow \{\eta\} \longrightarrow \{\eta\}^{\perp} \longrightarrow P(X,\eta)_2 \longrightarrow 0$$

where $^{\perp}$ means orthogonal complement with respect to the symmetric and antisymmetric bilinear form on $(JX)_2$ (see [I] p. 214 and [M2]).

The map P is generically surjective [B1] so as $\mathcal{P}_5$ is of dimension 12 and $\mathcal{A}_4$ of dimension 10 , the fibers of $P : \mathcal{P}_5 \to \mathcal{A}_4$ are generically of dimension 2 .

We call $\Sigma(X,\eta)$ (or $\Sigma(X)$ when $P(X,\eta)$ is fixed and there is no ambiguity on η) the surface image of the symmetric product $\tilde{X}^{(2)}$ by the map

$\tilde{X}^{(2)} \longrightarrow A$

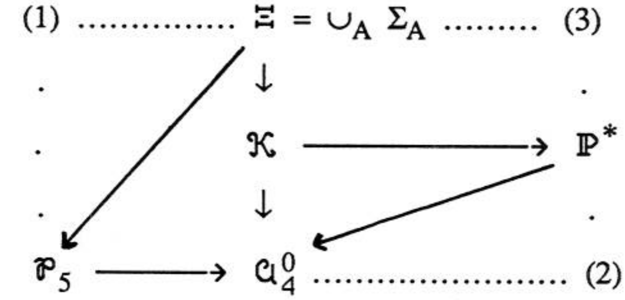
$(p,q) \longmapsto [p,q] = p+q-\sigma p-\sigma q$

(or for short $\Sigma(X)$ when $P(X,\eta)$ is fixed and there is no ambiguity on η). We show that the morphism

$$\Sigma_A = \bigcup_{(X,\eta)\in P^{-1}A} \Sigma(X,\eta) \longrightarrow A$$

is of degree 27 (outside the origin). Actually, if $\tilde{A}$ is the blow up of A at the origin, this morphism lifts to a morphism $\cup \tilde{X}^{(2)} \to \tilde{A}$.

So we have an induced morphism of degree 54 (outside the origin) from Σ_A to $K(A) = A_{/\pm id}$ and a commutative diagram :



We are going to fill in the "empty corners" of the above.

1.1 - The corner (1) of the diagram

For a generic curve X of genus 5 there is a net of quadrics containing its canonical model κX and a smooth plane quintic Q parametrizes the singular quadrics in the net. To each bisecant $\langle p,q\rangle$ of κX we associate the pencil l of quadrics containing $\langle p,q\rangle$ and κX.

So if we let $\widetilde{\mathcal{P}}_5$ be the moduli space of triples (X,η,l) where $(X,\eta) \in \mathcal{P}_5$ and l is a line in the net of quadrics containing κX, there is a map $\Xi \to \widetilde{\mathcal{P}}_5$ of degree 64 and we put $\widetilde{\mathcal{P}}_5$ in (1) with its canonical projection onto $\mathcal{P}_5$.

1.2 - The corner (2) of the diagram

Consider an abelian variety $A \in \mathcal{A}_4$, recall that $\widetilde{A}$ is its blow up at 0. We show that if A is neither in the closure $\overline{\mathcal{J}}_4$ of the locus $\mathcal{J}_4$ of jacobians of curves nor in a certain subvariety $\mathcal{A}_{n11}$ of $\mathcal{A}_4$ (see section 7), then the Γ_{00}-map extends to a *morphism* $\widetilde{h}$ from $\widetilde{A}$ to $(|2\Theta|_{00})^*$ of generic degree 2^7. The branch locus of $\widetilde{h}$ has 2 components :

- the image R_0 of the exceptional divisor
- the dual of a certain cubic threefold T in $|2\Theta|_{00}$ arising in a natural way :

Consider the moduli space $\mathcal{Z}$ of quartic double solids with six ordinary nodes. A double solid Z is a double cover of $\mathbb{P}^3$ branched along an even degree surface. A quartic double solid has as branch locus a quartic surface. When the quartic surface is smooth, the intermediate jacobian

$$JZ \;=\; H^{2,1}(Z)^* \,/\, H_3(Z)$$

is a ppav of dimension 10. When the branch locus acquires ordinary double points in general position the rank of JZ drops : 1 for each double point. So for a quartic double solid with 6 ordinary double points in general position, JZ is an element of $\mathcal{A}_4$.

The rational map $J : \mathcal{Z} \to \mathcal{A}_4$ is generically surjective [C1] hence its generic fibers have dimension 3.

To each $Z \in J^{-1}(A)$ Clemens associates an element D_Z of $|2\Theta|_{00}$ (see 2.1). We have a commutative diagram

$$\begin{array}{ccc} & D & \\ \mathcal{Z} & \to & \mathbb{P} \\ & J\searrow & \downarrow \\ & & \mathcal{A}_4^0 \end{array}$$

We show that the image $D(J^{-1}(A))$ is the cubic threefold T. The cubic threefold T comes with a point μ of order 2 in its intermediate jacobian.

Let $p_1,\ldots,p_6$ be the double points of a double solid Z. Let $\tilde{Z}$ be the blow up of Z at its double points. The threefold $\tilde{Z}$ admits 42 conic bundle structures over $\mathbb{P}^2$ which permit us to write $JZ = P(X,\eta)$ in 42 distinct ways. For 12 of these, say (X_i,η_i) , (X_i',η_i') for $i \in \{1,\ldots,6\}$, the double covers $\tilde{X}_i$ and $\tilde{X}_i'$ parametrize respectively lines through p_i and twisted cubics through p_j (for all $j \neq i$) in Z.

We show that there are embeddings of $\Sigma(X_i,\eta_i)$ and $\Sigma(X_i',\eta_i')$ in D_Z.

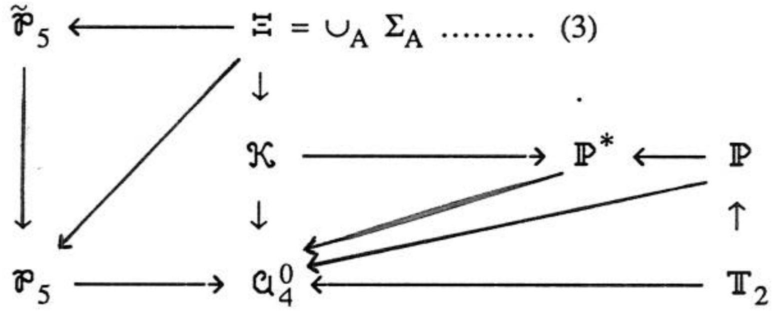
Fixing (X,η) there is a one dimensional family of double solids Z such that $(X,\eta) = (X_i,\eta_i)$ for some i, then (X_i',η_i') does not depend on the double solid Z in this family and is denoted by (X_λ,η_λ). By an argument of homology class we deduce that the one dimensional family of double solids gives us a pencil l_X of Γ_{00}-divisors D_Z in T containing $\Sigma(X,\eta)$ and $\Sigma(X_\lambda,\eta_\lambda)$.

Hence we obtain a 2-to-1 map $l : P^{-1}(A) \to F$, F being the Fano surface of lines of T. The involution acting in the fibers of this map is λ. This map and the involution λ were first introduced by Donagi in a different way.

The double cover $P^{-1}(A) \to F$ defines a point μ of order 2 in the albanese variety of F which is isomorphic to JT ([CG]). We will see below that μ is even with respect to the standard quadratic form on $(JT)_2$, where for an abelian variety B , B_2 denotes the set of points of order 2 .

This gives a birational correspondance (first defined abstractly by Donagi) between $\mathcal{A}_4$ and the moduli space $\mathcal{T}_2$ of cubic threefolds with an even point of order 2 in their intermediate jacobian.

Letting $\mathbb{P} \rightarrow \mathbb{P}^*$ be defined by the partial derivatives of the cubic threefolds T in each fiber and $\mathbb{T}_2$ be the universal cubic threefold over $\mathcal{T}_2$, we can fill the corner (2) of our diagram :



1.3 - The corner (3) of the diagram

We give an interpretation of the Γ_{00}-map $\tilde{h}$ in terms of Prym-embeddings of curves in intersections of two translates of Θ .

Let $(\tilde{X},X) = (X,\eta)$ be in $P^{-1}(A)$. The antisymmetric part of $J\tilde{X}$ for the involution σ induced by the covering involution on $\tilde{X}$ is the union of the "even part" A = $P(X,\eta)$ and "the odd part" A^- : a translate of A by an odd point of order 2 in $J\tilde{X}$ ([M2]). (Odd here means odd with respect to the standard $\mathbb{Z}/2\mathbb{Z}$-valued quadratic form on the points of order 2 in $J\tilde{X}$ (see [I] p. 214 and [M2]).)

The set $\{p-\sigma p ; p \in \tilde{X}\}$ is contained in A^- . A Prym-embedding of $\tilde{X}$ in A is by definition a translate of $\{p-\sigma p ; p \in \tilde{X}\}$ by an element of A^- .

For an element x of A , let t_x be translation by x in A , i.e., $t_x(y) = x+y$ for all $y \in A$. Let $\Theta_x = (t_{-x})^*\Theta$ be the translate of Θ by x . We show that for a generic element x of A the intersection $\Theta.\Theta_x$ contains exactly two Prym-embeddings of 27 distinct Prym curves for A and that the lines in T corresponding to these curves are the lines in the hyperplane section of T given by $\tilde{h}(x)$. This is equivalent to the fact that $\Sigma_A \to A$ has degree 27 .

At this point we need to introduce Donagi's tetragonal construction ([Do1]) :

For a curve C we denote by g^r_d the linear systems of degree d and projective dimension r on C . Consider a curve C of genus g with a g^1_4 and an etale double cover $\tilde{C}$. The curve $\tilde{C}$ is of genus $2g-1$. The set of liftings of divisors of g^1_4 in $\tilde{C}^{(4)}$ is a curve. This curve splits into two irreducible components $\tilde{C}'$ and $\tilde{C}''$: intuitively $\tilde{C}'$ parametrizes the liftings which have an even number of points from either sheet of the cover $\tilde{C} \to C$, and $\tilde{C}''$ parametrizes the liftings which have an odd number of points from either sheet of the cover. The curves $\tilde{C}'$ and $\tilde{C}''$ each come with an involution say σ' and σ'' . These involutions interchange complementary liftings of the same divisor in g^1_4 . The quotients $C' = \tilde{C}'/_{\sigma'}$ and $C'' = \tilde{C}''/_{\sigma''}$ have genus g , and both C' and C'' have g^1_4's : on C' for instance the points of a divisor of the g^1_4 are the classes modulo σ' of the "even" liftings of a given divisor of the g^1_4 on C .

Moreover the construction is symmetric.

The tetragonal construction of Donagi [Do1] and the trigonal construction of Recillas [R] are interesting special cases of a more general construction in [B3] .

For the double solid Z the curves (X_i,η_i) and (X_j,η_j) are tetragonally related whenever $i \neq j$. For fixed i and j the third curve (X_{ij},η_{ij}) in the tetragonal relation is a discriminant curve for one of the other 30 conic bundle structures of $\tilde{Z}$. More precisely $\tilde{X}_{ij}$ parametrizes pairs of incident lines in $\tilde{Z}$, one of which is an element of $\tilde{X}_i$ and the other an element of $\tilde{X}_j$. So 15 curves occur in this way.

Similarly (X_i,η_i) and (X'_j,η'_j) are tetragonally related whenever $i \neq j$. The third curves (X'_{ij},η'_{ij}) in the tetragonal relations give the other 15 conic bundle structures on $\tilde{Z}$.

We have $(X'_{ij},\eta'_{ij}) = \lambda\,(X_{ij},\eta_{ij})$. The 2 g^1_4's on X_i relating (X_i,η_i) to (X_j,η_j) and (X'_j,η'_j) are opposite, that is if we denote them by g and h then $g+h \equiv K_{X_i}$ where K_{X_i} is a canonical divisor on X_i ("$\equiv$" denotes linear equivalence of divisors).

Fix an element (X,η) of $P^{-1}(A)$. We denote by $\mathrm{Pic}^d X$ the principal homogeneous space on JX parametrizing linear systems of degree d on X. By a suitable isomorphism $JX \cong \mathrm{Pic}^4 X$, each g^1_4 on X goes to a singular point of the theta divisor Θ' of JX. The tangent cone to the theta divisor at a g^1_4 is a singular quadric containing κX. This defines a 2 to 1 map from the singular locus $\mathrm{Sing}\Theta'$ of Θ' onto the plane quintic Q. Under this map each g^1_4 and its opposite $K_X - g^1_4$ go to the same singular quadric q. A given q is of rank less than or equal to 4, if the rank is exactly 4 it has two rulings by planes. One ruling cuts on κX the divisors of g^1_4 and the other cuts on κX the divisors of $K_X - g^1_4$ [AM]. Also JX is the Prym variety of the double cover $\mathrm{Sing}\Theta' \to Q$.

Back to the threefold T and its family of lines, we reprove the fact that two Prym curves are tetragonally related if and only if their lines are incident. Originally Donagi proved this on the threefold T which he associated to A by an intricate indirect construction. Let l_X be the line in T associated to X. The projection from l_X

$$|2\Theta|_{00} \to \mathbb{P}^2$$

defines a conic bundle structure on T with discriminant curve Q. For a Γ_{00}-divisor D let q be the quadric in $|K_X|^*$ that is its image by the projection from l_X. We show that for each D, $D \cap \Sigma(X)$ is the set of points that give bisecant lines to κX that are contained in q.

The family of lines incident to l_X is a double cover $\tilde{Q}$ of Q with Prym variety the intermediate jacobian of T. The two points of $\tilde{Q}$ over a point q of Q correspond to the two lines in T in the plane section of T corresponding to q. The point α of order 2 on JQ corresponding to this double cover is odd.

From this we conclude that (X,η) and (X_λ,η_λ) have the same plane quintic Q. We obtain three points of order 2 in JQ for the three double covers of Q with Pryms JT, JX and JX_λ. We show that these points are the three elements of a rank 2 vector space on $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ contained in $(JQ)_2$. This subspace is totally isotropic with respect to the symmetric and antisymmetric bilinear form on $(JQ)_2$. (This is Donagi's starting point)

Let $\mathcal{G}$ be the moduli space of triples (Q,V^2,l) where Q is a plane quintic, V^2 is a rank 2 vector space on $\mathbb{F}_2$ contained in $(JQ)_2$ such that V^2 is totally isotropic with respect to the symmetric and antisymmetric bilinear form on $(JQ)_2$ and V^2 contains an odd point and two even points, l is a line in the plane of Q.

We define a map from Ξ to $\mathcal{G}$ in the following natural way :

Let $[p,q] \in \Sigma(X,\eta)$ ($(X,\eta) \in P^{-1}(A)$) then the image of $[p,q]$ is the triple (Q,V^2,l) where Q is the quintic parametrizing singular quadrics containing κX, V^2 is the inverse image of η by the last map in the exact sequence associated to the Prym construction $P(\mathrm{Sing}\Theta',Q) = JX$

$$0 \longrightarrow \{\alpha'\} \longrightarrow \{\alpha'\}^{\perp} \longrightarrow (JX)_2 \longrightarrow 0$$

and l is the pencil of quadrics containg κX and the line $\langle p,q\rangle$ in the canonical space of X.

Given (Q,V^2,l) we get the abelian variety A from (Q,V^2) by two successive Prym constructions, and a hyperplane section of $|2\Theta|_{00}$ from l. Hence a map from $\mathcal{G}$ to $\mathbb{P}^*$ of degree 27. We can now complete our diagram :

$$(*) \qquad \begin{array}{ccccccc}
\tilde{\mathcal{P}}_5 & \longrightarrow & \Xi = \cup_A \Sigma_A & \longrightarrow & \mathcal{G} & & \\
\downarrow & \swarrow & \downarrow & & \downarrow & & \\
 & & \mathcal{K} & \longrightarrow & \mathbb{P}^* & \longleftarrow & \mathbb{P} \\
\downarrow & & \downarrow & \swarrow & & \swarrow & \uparrow \\
\mathcal{P}_5 & \longrightarrow & \mathcal{A}_4 & & \longleftarrow & & \mathbb{T}_2
\end{array}$$

1.4 - Relation of the diagram with $\mathcal{A}_5$ and $\mathcal{M}_6$

We let $\mathcal{M}_g$ be the moduli space of curves of genus g. For all families of ppav's of dimension g parametrized by a variety M we denote by $\overline{M}$ the partial compactification of M by $\mathbb{C}^*$-extensions of (g-1)-dimensional ppav's with properties similar to those of elements of M. We denote by M^0 (resp. $\overline{M}^0$) the open subset of "automorphism free" group schemes.

Then $\mathcal{K} \subset \overline{\mathcal{A}}_5^0$ is the boundary :

$$\mathcal{K} = \overline{\mathcal{A}}_5^0 \setminus \mathcal{A}_5^0$$

because if we let G_m be the multiplicative group then we have ([M1], page 227) :

$$A \cong \mathrm{Pic}^0 A \cong \mathrm{Ext}^1(A, G_m)$$

where $\mathrm{Pic}^0 A$ is the dual abelian variety of A or the abelian variety parametrizing homologically trivial line bundles on A and $\mathrm{Ext}^1(A,G_m)$ is the group of $(G_m(\mathbb{C}) = \mathbb{C}^*)$-extensions of A. Two extensions of A by $\mathbb{C}^*$ with extension data's b, $b' \in A$ are isomorphic if and only if there exists an automorphism ν of A which is not a translation such that $\nu(b) = b'$. So for $A \in \mathcal{A}_4^0$ the isomorphism classes of $\mathbb{C}^*$-extensions of A are parametrized by $K(A) = A_{/\pm id}$.

Let $\mathcal{A}_{g,2}$ denote the moduli space of ppav's of dimension g with a nontrivial point of order 2. Let $\tilde{\mathcal{A}}_{5,2}$ (resp. $\tilde{\mathcal{A}}_5$) be the blow up of $\overline{\mathcal{A}}_{5,2}$ (resp. $\overline{\mathcal{A}}_5$) along the loci $\mathcal{P}_5$ (resp. $\mathcal{J}_5$ of jacobians) and $\mathcal{T}_2$ (resp. $\mathcal{T}$) of intermediate jacobians of cubic threefolds.

We can relate all these maps and spaces to our diagram (*) as follows :

1) $\tilde{\mathcal{P}}_5 \subset \tilde{\mathcal{A}}_{5,2}$ is the strict transform of $\mathcal{P}_5$. This is because the conormal space to the jacobian locus at a jacobian JX can be canonically identified with I (the vector space of quadrics containing κX) by the exact sequence

$$0 \to I \to S^2H^0(X,\omega_X) \to H^0(X,(\omega_X)^2) \to 0$$

(here ω_X is the canonical sheaf of X).

2) $\mathbb{P}^* \subset \tilde{\mathcal{A}}_{5,2}$ is the strict transform of the locus $\mathcal{T}'_2 \subset \mathcal{T}_2$ of intermediate jacobians of cubic threefolds with an *even* point of order 2 . (This is because of the fact that the conormal space of $\mathcal{T}$ in $\tilde{\mathcal{A}}_5$ at a cubic threefold T can be canonically identified with the ambiant $\mathbb{P}^4$ of T [DS] .)

3) If $\mathcal{P}_6^2$ is the moduli space of curves C of genus 6 with a totally isotropic subspace of rank 2 in $(JC)_2$ (with two even points of order 2 and one odd point of order 2) and we let $\tilde{\mathcal{P}}_6^2$ be the blow up of $\mathcal{P}_6^2$ along the locus of plane quintics, then we have $\mathcal{Q} \subset \tilde{\mathcal{P}}_6^2$. This follows from the fact that the conormal space to the locus of plane quintics in $\mathcal{M}_6$ at a point Q can be canonically identified with the ambiant $\mathbb{P}^2$ of Q [DS].

4) The map $\mathcal{Q} \to \mathbb{P}^*$ in our diagram is induced by the Prym map.

5) Let $\tilde{\mathcal{P}}_5$ project onto $\tilde{\mathcal{M}}_5 \cong \tilde{\mathcal{J}}_5$ in $\tilde{\mathcal{A}}_5$ and $\mathbb{P}^*$ project onto $(\mathbb{P}')^*$, then we have a diagram of correspondances between subvarieties of $\tilde{\mathcal{A}}_5$, all dominated by Ξ :

$$\begin{array}{ccccc} & & \Xi & & \\ & \swarrow & \downarrow & \searrow & \\ \tilde{\mathcal{M}}_5 & & \mathcal{K} & \to & (\mathbb{P}')^* \end{array}$$

1.5 - Prym curves over the boundary of $\mathcal{A}_5$ and $\Theta.\Theta_x$ in $A \in \mathcal{A}_4$

(1.5.1) By [DS] the degree of the Prym map (still denoted by P) from $\mathcal{P}_6$ to $\mathcal{A}_5$ is 27 .

Recall that $\mathcal{K} = \tilde{\mathfrak{a}}_5 \setminus \mathfrak{a}_5$ is the boundary. The inverse image of $\mathcal{K}$ under the extension of P to the moduli space of stable curves with a point of order 2 is the variety $\mathcal{G}en\mathcal{P}_6$ of generalized jacobians of singular irreducible curves with one node and a double cover with two nodes exchanged by the covering involution.

For each Prym-curve $\tilde{X}$ in $\Theta.\Theta_x$ write $x = [p,q] \in \Sigma(X,\eta)$, then the extension A_x of A corresponding to x is the Prym variety of the double cover $\tilde{X}_{pq} = \tilde{X}_{/p=\sigma q, \sigma p=q}$ of the curve $X_{pq} = X_{/\pi p = \pi q}$. We show :

The extension of P *to* $\mathcal{G}en\mathcal{P}_6 \cup \mathcal{P}_6$ *is generically unramified on* $\mathcal{G}en\mathcal{P}_6$.

So using the properness of P [B1], we obtain a second proof of the result of [DS] by combining the theorem with the fact that a generic intersection of two translates of Θ in A contains 27 Prym curves.

(1.5.2) Let B be an element of $\tilde{\mathfrak{a}}_5$. The tetragonal relation defines a correspondance between the 27 elements of $P^{-1}(B)$ which is similar to the incidence correspondence among the lines in a cubic surface in $\mathbb{P}^3$. A natural question is :

Can one produce a family of cubic surfaces on $\tilde{\mathfrak{a}}_5$ and an $\tilde{\mathfrak{a}}_5$-isomorphism between $\tilde{\mathcal{P}}_5$ and the family of lines in these surfaces ?

Such a family is easily produced generically on the strict transform $(\mathbb{P}')^* \cup (\mathbb{P}'')^*$ of $\mathfrak{T}$: an element $B \in (\mathbb{P}')^* \cup (\mathbb{P}'')^*$ consists of a pair (T,n) where $T \in \mathfrak{T}$ and n is a normal direction to $\mathfrak{T}$ in $\mathfrak{a}_5$ hence n defines a hyperplane section of T which is the desired cubic surface.

We can produce this family on $\mathcal{K}$: an element B of $\mathcal{K}$ is an extension of a four-dimensional ppav A with extension data $b \in A$. Then $\tilde{h}(b) \in (|2\Theta|_{00})^*$ gives a hyperplane section of the cubic threefold T associated to A which is the cubic surface we are looking for.

A family of cubic surfaces cannot be produced on $\tilde{\mathcal{J}}_5$ but instead on $\tilde{\mathcal{P}}_5$: let $(X,\eta,l) \in \tilde{\mathcal{P}}_5$, then by inverse image in the exact sequence

$$0 \to \{\alpha'\} \to \{\alpha'\}^{\perp} \to (JX)_2 \to 0$$

associated to the Prym construction $P(\mathrm{Sing}\Theta',Q) = P(Q,\alpha') = JX$ one gets as before a rank 2 totally isotropic subspace of $(JQ)_2$ and an odd point α of order 2 in this subspace (see 5.29). Hence $P(Q,\alpha) = JT$ for a cubic threefold T. We also have a projection

$$\begin{array}{ccc} T & \subset & \mathbb{P}^4 \\ & \searrow & \downarrow \\ & & \mathbb{P}^2 \supset Q \end{array}$$

from a line $l(Q) = l_X$ in T such that $T \to \mathbb{P}^2$ is a conic bundle with discriminant curve Q. The inverse image of l in $\mathbb{P}^4$ by this projection gives the required hyperplane section of T.

This suggests that maybe one should look for the family of cubic surfaces over a level moduli space $\widetilde{\mathcal{A}}_{5,i}$ $(i \geq 2)$. One might be able to produce it over $\widetilde{\mathcal{A}}_{5,2}$ or $\widetilde{\mathcal{P}}_6^2$.

1.6 - The branch locus of P in $\mathcal{A}_5$ and the branch locus of $\tilde{h}$

We need to introduce the Abel-Jacobi mapping AJ and the Fano variety (intersection of two translates of Θ) associated to a double solid.

The intermediate jacobian JZ can be identified with the group of algebraic one-cycles in $\tilde{Z}$, which are algebraically equivalent to 0, modulo rational equivalence. Fixing a curve C_0 of degree d in $\tilde{Z}$ we define (up to translation) the Abel-Jacobi mapping AJ on the Hilbert Scheme of curves of degree d in $\tilde{Z}$ by sending a curve C to the rational equivalence class of $C - C_0$.

Alternatively we can define $AJ(C)$ to be the linear form on $H^{2,1}(Z)$ which to a differential form ω associates $\int_\Omega \omega$ where Ω is an $\mathbb{R}$-three-dimensional singular chain such that its boundary $\partial\Omega$ is equal to $C - C_0$ (this is well-defined modulo $H_3(Z,\mathbb{Z})$).

Let E_Z be the image under AJ of the Fano variety of lines of Z. Using the computation of the homology class of E_Z in A in [C1], Beauville shows that $E_Z = \Theta.\Theta_x$ for an $x \in \cap \Sigma(X_i,\eta_i)$ (see the beginning of the introduction).

Recall that $\tilde{h}$ is the lift of the Γ_{00}-map to the blow up $\tilde{A}$ of A at 0. Let E be the exceptional divisor in $\tilde{A}$. We have the following description of the branch locus of $\tilde{h}$:

The branch locus of $\tilde{h}$ *in* $(\mathbb{P}^4)^*$ *is the union of the dual hypersurface* T^* *of* T *and an irreducible hypersurface* $R_0 = \tilde{h}(E)$ *of* $(|2\Theta|_{00})^*$ *whose inverse image* R *in* A *is the union of the diagonals of the surfaces* $\Sigma(X)$ *for* $X \in P^{-1}(A)$.

Moreover if $\mathcal{E}$ *is the variety of* $a \in A$ *such that* $\Theta.\Theta_a$ *is the fano variety of lines* E_Z *of a double solid* Z *then the inverse image of* T^* *in* A *is the union of* $\mathcal{E}$ *and another irreducible component* R' *whose fibers over* T^* *have cardinality* 64. *The inverse image of a generic element of* T^* *has cardinality* 96. *For* $a \in R'$, $\Theta.\Theta_a$ *contains* 21 *curves,* 6 *of them counting twice.*

From this we derive a second proof (there is an unpublished proof of this in [Do3]) of the fact that the branch locus of P in $\mathcal{A}_5$ is the variety of intermediate jacobians of double solids ; showing that extensions A_x with $x \in R'$ are generalized intermediate jacobians of double solids.

1.7 - Schottky

The Schottky problem is the problem of characterizing jacobians in $\mathcal{A}_4$. In relation with that we prove :

If A *is not in* $\overline{\mathcal{J}}_4 \cup \mathcal{A}_{n11}$ *then the only base point of* $|2\Theta|_{00}$ *is* 0 *with multiplicity* 2^8.

Here $\mathcal{J}_4$ is the locus of jacobians, $\overline{\mathcal{J}}_4$ is its closure in $\mathcal{A}_4$, and $\mathcal{A}_{n11}$ is a six-dimensional subvariety of $\mathcal{A}_4$ which we will define later (see section 7).

Let $V(\Gamma_{00})$ be the base locus of $|2\Theta|_{00}$. Let $\mathcal{A}_{4dec}$ be the locus of decomposable ppav's. Together with the result of [W3] this gives :

Let A *be in* $\mathcal{A}_4 \setminus (\mathcal{A}_{n11} \cup \mathcal{A}_{4dec})$ *then* A *is a jacobian if and only if* $V(\Gamma_{00})$ *contains another point besides* 0 , *in that case* $V(\Gamma_{00}) = C\text{-}C \cup \{\pm(K_C - 2g^1_3)\}$ (C-C = {p-q ; p,q ∈ C}).

From the computation of the multiplicity (in a sense which will be made precise later) at 0 of $|2\Theta|_{00}$ we deduce :

If A *is not in* $\overline{\mathcal{J}}_4 \cup \mathcal{A}_{n11}$, *then the linear system* $\tau|2\Theta|_{00}$ *of quartics in* $\mathbb{P}^3 = \mathbb{P}T_0A$ *obtained by taking the quartic tangent cones at* 0 *of the elements of* $|2\Theta|_{00}$ *has (projective) dimension at least* 3 *and its base locus* $V_{inf}(\Gamma_{00})$ *is empty.*

Together with the result of [BD2], this gives :

Let A *be in* $\mathcal{A}_4 \setminus (\mathcal{A}_{n11} \cup \mathcal{A}_{4dec})$ *then* A *is a jacobian if and only if* $V_{inf}(\Gamma_{00})$ *is nonempty. In that case, if the unique quadric containing the canonical curve is smooth,* $V_{inf}(\Gamma_{00})$ *is the canonical curve or, if the quadric containing the canonical curve is not smooth,* $V_{inf}(\Gamma_{00})$ *is the union of the canonical curve and the vertex of the quadric containing it.*

1.8 - Jacobians, one theta-null and singular cubic threefolds

Using the (generic) description of the cubic threefold as the hypersurface containing pencils of Γ_{00}-divisors associated to Prym curves and as the dual of a well-specified component of the branch locus of the Γ_{00}-map we can see that the cubic threefold can be defined for all elements of $\mathcal{A}_4 \setminus (\overline{\mathcal{J}}_4 \cup \mathcal{A}_{n11})$ and also for jacobians of irreducible nonhyperelliptic curves.

(1.8.1) So the question arises : for which abelian varieties is the cubic threefold singular ? The answer is

The cubic threefold is singular exactly on the locus $\mathcal{J}_4$ *of jacobians and on the divisor* θ_{null} *of ppav's with one vanishing theta-null, i.e., those abelian varieties whose* *theta divisor contains a point of order 2 which is then a singular point of* Θ.

When the cubic threefold T has one ordinary double point and is otherwise smooth the intermediate jacobian of T is the jacobian of a nonhyperelliptic curve C of genus 4 [CG]. The generalized intermediate jacobian $\tilde{J}T$ of T is an extension of JT by $\mathbb{C}^*$ ($\tilde{J}T$ is not the generalized jacobian of a curve by [Co]). The point μ of order 2 on $\tilde{J}T$ corresponding to JC projects to 0 in JT. Hence μ is the only nontrivial point of order 2 in $\mathbb{C}^* \subset \tilde{J}T$. The even points of order 2 on $\tilde{J}T$ which occur for ppav's with one vanishing theta-null do not project to 0 in JT.

We have a finite rational map

$$\theta_{null} \rightarrow \mathfrak{I}_4$$

By the above the degree of this map is

$$2^4(2^3+1) - 1 + 2^4(2^3+1) - 1 .$$

(1.8.2) What happens to the family of double solids $\mathcal{Z}_A$ with intermediate jacobian a fixed ppav A when the cubic threefold becomes singular ?

On θ_{null} the family $\mathcal{Z}_A$ breaks up into two components : one of them, say $(\mathcal{Z}_A)_0$, is blown down to the singular point of T and the other, say $(\mathcal{Z}_A)_1$, dominates T.

When T has one ordinary double point and is otherwise smooth the Fano variety of lines F of T has a double curve which parametrizes lines through the double point of T, the desingularization of F is the second symmetric product $C^{(2)}$ of C [CG]. The one-dimensional component of the branch locus of the double cover $P^{-1}(A) \rightarrow F$ is the double curve of F. The component $(\mathcal{Z}_A)_0$ parametrizes double solids whose discriminant curves verify $(X_i,\eta_i) = \lambda(X_i,\eta_i) = (X_i',\eta_i')$.

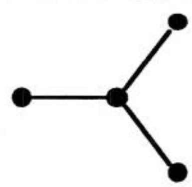
It is proved in [C1] that when the branch locus of the double solid is a quadric in the quadrics through the double points $p_1,\ldots,p_6$ of Z the intermediate jacobian JZ = A has a vanishing theta-null. These are precisely the elements of $(\mathcal{Z}_A)_0$.

The component $(\mathcal{Z}_A)_1$ parametrizes double solids with determinantal branch locus.

In the jacobian case Donagi proves [Do2] that the intermediate jacobians of double solids Z with "unodal" branch locus are jacobians. A unode counts as three ordinary double points and has local equation

$$x^2 + (y - az).(y - bz).(y - cz) + \text{arbitrary higher order} = 0$$

The tangent cone at a unode is twice a line and the cubic term of the equation corresponds to three infinitely near double points in that line. The incidence configuration of the exceptional curves in the resolution of a unode is given by the Dynkin diagram D_4 :



It is easy to see that the converse is true, i.e., when JZ is a jacobian then the branch locus of Z has a unode. Double solids with two unodes correspond to elements of $\theta_{null} \cap \mathcal{J}_4$.

In the jacobian case, there are no double solids above the singular point t of T .

1.9 - Torelli for quartic double solids with six nodes

We are going to describe a geometric way of recovering each element of $\mathcal{Z}_A$ (see 1.7) from the abelian variety A . Suppose A generic in $\mathcal{A}_4$.

(1.9.1) Recall that we have a finite surjective map $\mathcal{Z}_A \to T$ where (T,μ) is the cubic threefold with point of order 2 associated to A . The degree of this map is 32 and we describe its fibers below (Donagi first proved this on the threefold T which he associated to A in a different way : we will write down his construction later).

Let $Z \in \mathcal{Z}_A$ and denote by $p_1,\ldots,p_6$ its double points. Let $\rho : \mathbb{P}^3 \to \mathbb{P}^3$ be the rational map of degree 2 defined by the linear system of quadrics in $\mathbb{P}^3$ containing the

p_i's . Denote by ι the birational involution on $\mathbb{P}^3$ commuting with ρ . Also for each set $\mathfrak{F}$ of cardinality 4 contained in $\{p_1,\ldots,p_6\}$ we denote by $\iota_{\mathfrak{F}}$ the birational involution of $\mathbb{P}^3$ defined by the linear system of cubics in $\mathbb{P}^3$ with double points at the elements of $\mathfrak{F}$. These are Cremona transformations of $\mathbb{P}^3$ (see [DO]). Hence we can talk about the subgroup $\mathcal{G}$ of the Cremona group of $\mathbb{P}^3$ generated by $\{ \iota_{\mathfrak{F}} \}_{\mathfrak{F}}$. The group $\mathcal{G}$ contains the birational involution ι and is isomophic to $(\mathbb{Z}/2\mathbb{Z})^5$.

If B is the branch locus of $Z \to \mathbb{P}^3$, then we denote by ιZ , $\iota_{\mathfrak{F}} Z$ the double solids with respective branch loci ιB , $\iota_{\mathfrak{F}} B$. These double solids are all birationally equivalent and have the same intermediate jacobian A .

The double solids Z and ιZ clearly have the same double points. Also, for all $\mathfrak{F}$, the double points of $\iota_{\mathfrak{F}} Z$ have the same moduli in $\mathbb{P}^3$ as the p_i's . Letting $\mathcal{G}$ act on Z , we obtain a set of 32 double solids which breaks naturally into the union of 16 subsets $\{ Z_i , \iota Z_i \}$ ($1 \leq i \leq 16$) with, for instance, $Z = Z_1$ (this is not canonical and depends on the choice of Z).

We show that the set of double solids Z' such that $D_Z = D_{Z'}$ is equal to $\{ Z_i , \iota Z_i ; 1 \leq i \leq 16 \}$.

(1.9.2) Next, the moduli space of six points in $\mathbb{P}^3$ is birationally equivalent to the moduli space of six points in $\mathbb{P}^1$ which is birationally equivalent to the moduli space $\mathfrak{M}_2$ of curves of genus 2 via the following :

Through 6 generic points in $\mathbb{P}^3$ there passes a unique twisted cubic curve so we obtain 6 points in $\mathbb{P}^1$ by identifying the cubic with $\mathbb{P}^1$. To the 6 points in $\mathbb{P}^1$ there is associated a unique (up to isomorphism) curve M of genus 2 which is the double cover of the projective line ramified at the six points. The Kummer variety $K = JM_{/\pm id}$ is the branch locus of ρ .

As the tangent space to A at 0 can be canonically identified with

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}}(2) \otimes \mathcal{I})^*$$

where $\mathcal{J}$ is the ideal sheaf of the points p_i [C1], K lives in the projectivised tangent space $\mathbb{P}T_0A$. The 16 double points of K are the images by ρ of

- the 15 lines through p_i and p_j
- the twisted cubic $C_Z = C_{\iota Z}$ through $p_1,\ldots,p_6$

Each Z_i picks a double point of K. If $i \neq 1$ this is the image of the line $\langle p_j, p_k \rangle$ if $Z_i = \iota_{\mathfrak{F}} Z$ or $\iota_{\mathfrak{F}} \iota Z$ with

$$\mathfrak{F} = \{p_1,\ldots,p_6\} \setminus \{p_j, p_k\}$$

Under $\iota_{\mathfrak{F}}$ the line $\langle p_j, p_k \rangle$ goes to C_{Z_i}. The double point associated to $Z = Z_1$ is the image of C_Z.

(1.9.3) It is proven in [C2] that the projectivised tangent cone at 0 to D_Z is $\rho(B)$ (see 2.1).

Let $\tilde{\mathbb{P}}^3$ be the blow up of $\mathbb{P}^3$ at the points $p_1,\ldots,p_6$. Under the lift of ρ to $\tilde{\mathbb{P}}^3$, the images of the six exceptional planes are six planes P_i in $\mathbb{P}T_0A$. The double point corresponding to C_Z is contained in all the P_i's. The intersection $P_i \cap P_j$ also contains the double point $\rho(\langle p_i, p_j \rangle)$ for $i \neq j$. Each P_i contains six double points of K that are on a conic. The moduli of the six points on the conic is equal to the moduli of the points $p_i,\ldots,p_6$. Hence M is the double cover of the conic branched along the double points of K.

Each P_i contains two other distinguished conics : these are the images of the projectivised tangent cones at p_i to B and ιB. The intersection $\rho(B) \cap P_i$ is the union of these conics.

As $D_Z = D_{Z'}$ for all $Z' \in \{ Z_i , \iota Z_i ; 1 \leq i \leq 16 \}$, we have $\rho(B) = \rho(\iota B) = \rho(B_i) = \rho(\iota B_i)$ for all i.

From the above discussion we deduce that there are 16 well-determined positions of K with respect to $\rho(B)$ which come from the double solids. As $\rho : \mathbb{P}^3 \to \mathbb{P}^3$ is the double cover branched along K, once we are given K and D_Z (hence $\rho(B)$), we can construct the 32 double solids Z_i, ιZ_i.

Also, as the moduli of six double points of K on a conic gives the isomorphism class of M, K is determined by its double points.

All we need to do now is, given D_Z, find the double points of K. This is done below.

(1.9.4) We show that for each i, the images under AJ of the Fano varieties of lines of Z_i and ιZ_i are equal (up to translation and multiplication by -1), we denote them by E_i. For $i \neq j$, E_i and E_j are not images of each other by $\pm$id or any translation.

Generically, T is birationally equivalent to its dual T*. For $Z \in \mathcal{Z}_A$, the Γ_{00}-divisor $D_Z \in T$ associated to Z corresponds to a hyperplane H_Z in $(|2\Theta|_{00})^*$, tangent to T* at a point t_Z. The inverse image of H_Z in $\tilde{A}$ in by $\tilde{h}$ is, of course (the strict transform of) D_Z.

Let $\pm x_i$ ($1 \leq i \leq 16$) be the elements of A such that $\Theta.\Theta_{x_i} = E_i$ (see above).

As we saw in 1.2, T* is one component of the branch locus of $\tilde{h}$. By 1.6, above t_Z, $\tilde{h}$ is ramified exactly at the points $\pm x_i$ ($1 \leq i \leq 16$). We show that (after identification of $T_{\pm x_i}A$ with T_0A by translation by $\pm x_i$) the kernel of the differential of $\tilde{h}$ at $\pm x_i$ is the double point of K corresponding to Z_i.

1.10 - Theta-identities

We will also draw theta identities between an element of Γ_{00} defining D_Z and three elements of Γ determined by the Fano veriety of lines in Z.

2. GENERALITIES AND BACKGROUND

Here we write down some unpublished results of Beauville, Clemens, Debarre and Donagi.

2.1 - The Γ_{00}-divisor D_Z associated to a quartic double solid Z

We write down the construction of Clemens [C3] :

Let $\tilde{\pi}$ and π be the projections $\tilde{Z} \to \mathbb{P}^3$ and $Z \to \mathbb{P}^3$. Consider the Hilbert scheme of irreducible conics in $\tilde{Z}$, i.e., curves C in $\tilde{Z}$ such that

- $C.\tilde{\pi}^*\mathcal{O}_{\mathbb{P}^3}(1) = 2$
- $C.\mathcal{Q}_i = 0$ for all i, where $\mathcal{Q}_i$ is the exceptional quadric in $\tilde{Z}$ above p_i.

This Hilbert scheme has dimension 4 and has two components say $\mathcal{D}_Z$ and $\mathcal{D}'_Z$ The generic element of $\mathcal{D}'_Z$ is the inverse image of a line in $\mathbb{P}^3$, hence it is a smooth elliptic curve. The generic element of $\mathcal{D}_Z$ is one component of the inverse image in $\tilde{Z}$ of a conic in $\mathbb{P}^3$ which is everywhere tangent to the branch locus B of Z. So the generic element of $\mathcal{D}_Z$ is a smooth rational curve. Clemens defines $D_Z = AJ(\mathcal{D}_Z)$, where $AJ : \mathcal{D}_Z \to JZ$ is the Abel-Jacobi map with base curve the inverse image in $\tilde{Z}$ of a line in $\mathbb{P}^3$ which is tangent to B (The family of such conics is blown down to a point by AJ).

(2.1.1) **THEOREM** : *The divisor* D_Z *is an element of* $|2\Theta|_{00}$. *The tangent cone to* D_Z *at* 0 *is* $\rho(B)$: *the image of* B *by the map* ρ *given by the linear system of quadrics through the double points of* Z.

Proof : Let C be a conic in $\mathbb{P}^3$, everywhere tangent to B and generic for this property. Let V be the plane containing C. The data of C everywhere tangent to the canonical curve $S = B \cap V$ of genus 3 is equivalent to the data of a divisor class on S with square $(K_S)^2$ or equivalent to the data of a point γ of order 2 on JS. Then $|K_S+\gamma|$ is a pencil. Thus there is at least a pencil of curves rationally equivalent to C in $\tilde{Z}$. We want to find the reducible curves, i.e., the unions of two lines (bitangent to S) in this pencil. By [DO], odd theta-characteristics on S are in one-to-one correspondance with bitangent lines to S in V. A simple computation shows that there are 12 theta-characteristics k on S such that k and $k+\gamma$ is odd. Thus there are $6 = 12/2$ pairs of bitangent lines to S in V whose unions cut divisors of $|K_S+\gamma|$ on S, equivalently, there are six pairs of *incident* lines in $\tilde{Z}$ whose sums are rationally equivalent to C.

It follows in particular that D_Z is of dimension ≤ 3. The following implies that D_Z cannot have dimension less than 3.

Recall that E_Z is the Fano variety of lines of $\tilde{Z}$. We just saw that if $I \subset E_Z \times E_Z$ is the divisor of pairs of incident lines, then $AJ(I) = D_Z$.

Suppose, momentarily, that Z *is smooth, then* $A = JZ$ *is a ppav of dimension* 10.

By [W1] (p. 77) if

$$\phi : E_Z \times E_Z \to A$$

is induced by addition, then $\phi^*\Theta$ is homologous to I modulo $\mathrm{Pic}E_Z \oplus \mathrm{Pic}E_Z$. By [W1] (page 70) the restrictions of $\phi^*\Theta$ to two fibers $l \times E_Z$ and $E_Z \times l$ have homology class $3D_l$, where $D_l = I.(l \times E_Z)$ or $(E_Z \times l).I$ respectively. The D_l all have the same homology class D in E_Z. Hence $\phi^*\Theta$ is homologous to

$$I + 2(D \times E_Z) + 2(E_Z \times D)$$

A straightforward computation, using Pontrjagin product and

(A) $$\Theta^{(d-j)}/(d-j)! = \Sigma\, \gamma_{i_1} \times \delta_{i_1} \ldots \times \gamma_{i_j} \times \delta_{i_j}$$

(where j is any number between 1 and $d = \dim A = 10$ in this case , γ_i , δ_i form a symplectic basis of $H_1(A,\mathbb{Z})$, $\{i_1,\ldots,i_j\}$ is any set of distinct integers between 1 and d and "×" is Pontrjagin product) shows then that the homology class of ϕ_*I in A is $25.\Theta^7/_{7!}$. The map $I \to AJ(I)$ is of degree at least 12 . So the reduced image of I has homology class $n.\Theta^7/_{7!}$ where $n = 1$ or 2 . Now, using [BC] and degenerating one double point at a time to the case where Z has 6 double points, one sees that D_Z has homology class $n.\Theta$ where $n = 1$ or 2 .

At this point, Clemens observes that the tangent cone to D_Z at 0 is $\rho(B)$ (we give a proof below). So D_Z has a singularity of order ≥ 4 at the origin. As A is generic and Θ nonsingular , n has to be 2 (since, by 3.6, $|2\Theta|_{00}$ has no member with a singularity of order > 4 at 0 , the singularity of D_Z is of order exactly 4). That is $D_Z \in |2\Theta|_{00}$. (In the case where Z has less than six double points also $n = 2$ and the tangent cone at 0 to D_Z is $\rho(B)$.) The proof is as follows :

Let C and V be as before. Let C' and C" be the two components of the inverse image of C in $\tilde{V} = \tilde{\pi}^{-1}(V)$. Let |C'| be the linear system of C' in $\tilde{V}$. Using the identification $\mathbb{P}T_0A \cong |\mathcal{O}_{\mathbb{P}}(2)\otimes \mathcal{J}|^*$ (see 1.9), the projectivised tangent space to D_Z at C' corresponds to the unique quadric Q through the double points of Z such that either

$$N \subset \tilde{\pi}^*Q \quad \text{for some } N \in |C'|$$

or

$$N.\tilde{\pi}^*Q = N.B \quad \text{for some } N \in |C'|$$

here $\tilde{\pi} : \tilde{Z} \to \mathbb{P}^3$ is the projection. Let B' be an element of the pencil spanned by B and $\tilde{\pi}^*Q$ which contains some element N of |C'| . Let l be a line in $\mathbb{P}^3$, tangent to B at a point t . The curve $\tilde{\pi}^*l$ has an embedded point at t . Going to the limit, one sees that when $N = \tilde{\pi}^*l$, B' has a singular point at t . Hence, B and Q are tangent at t . Hence, $\rho(B)$ and the limits (at 0) of tangent planes to D_Z are tangent. Hence, $\rho(B)$ is the projectivised tangent cone at 0 to D_Z .

2.2 - How to recover B from the plane representation of X_i

The curves X_i (see 1.2) are also discriminant curves for the projections of B from p_i. For all i, X_i is a plane sextic with five double points corresponding to p_j for $j \neq i$. Each plane sextic X_i admits an everywhere tangent conic C_{ti} : this is the image of the projectivised tangent cone to B at p_i under r_i. Here r_i is the extension of the projection from p_i to the blow up $\mathbb{P}^3_i$ of $\mathbb{P}^3$ at p_i. The six points of contact of C_{ti} with X_i are the images by r_i of the six lines through p_i which have contact of order 4 with B at p_i. A result of Donagi is [Do2] :

(2.3.1) **THEOREM** : *Let* B *be a generic quartic in* $\mathbb{P}^3$ *with six double points in general position. Then the plane representation of* X_i *determines the conic* C_{ti} *which, together with the plane representation, determines* B , *hence also* Z .

Proof : One first proves the uniqueness of the everywhere tangent conic.

Let $\tilde{B}$ be the blow up of B at its double points. Notice that $\tilde{B}$ is the minimal desingularization of B and that $\tilde{B}$ is a K3 surface. Donagi proves

(2.3.2) **LEMMA** : *Generically, the Picard number of* $\tilde{B}$ *is* 6 . *The Picard group of* $\tilde{B}$ *is generated (over* $\mathbb{Z}$ *) by the exceptional curves* E_i *above* p_i *and the inverse image* H *of the hyperplane class in* $\mathbb{P}^2$. *These verify (for all* i *and* $j \neq i$ *)*

$$H^2 = 4 \,,\ (E_i)^2 = -2 \,,\ H.E_i = E_i.E_j = 0$$

Assume the lemma for a moment. Suppose that there exists another everywhere tangent conic to X_1, say C . The inverse image of C in $\tilde{B}$ has two components, say C' and C''. Let $p : \tilde{B} \to \mathbb{P}^2$ be the projection. The following numerical computations yield a contradiction :

a) $C'^2 = -2$ by adjunction.

b) $C'.(H - E_1) = 2$

This is because $H - E_1$ is the line bundle associated to p and $C'.(H - E_1)$ is the degree of $p_*C' = C$ in $\mathbb{P}^2$.

c) $0 \leq E_2.C' \leq 4$

Because both curves lie over distinct conics in $\mathbb{P}^2$.

d) $C'.E_i \geq 0$ for all i.

By the lemma we can write

$$C' = aH + \Sigma b_j E_j \quad \text{for some } a, b_j \in \mathbb{Z}$$

then the above become

a) $4a^2 - 3.\Sigma b_j^2 = -2$

b) $4a + 2b_1 = 2$

c) $-2 \leq b_1 \leq 0$

d) $b_i \leq 0$ for all i.

From b) and c) one obtains $a = 1$ and $b_1 = -1$. Then by a) exactly two of the b_j's are nonzero for $j \neq 1$, say 2 and 3, and $b_2^2 = b_3^2 = 1$. Hence $C' = H - E_1 - E_2 - E_3$ and p_*C' is twice a line.

Proof of the lemma : The assertion about the intersection numbers of H and the E_i's is clear. By [BPV] Chapter 8, for every K3 surface B, $H^2(B,\mathbb{Z})$ equipped with the quadratic form given by cup-product is isomophic to the lattice

$$L = -E_8 \oplus -E_8 \oplus H \oplus H \oplus H$$

where E_8 is the free $\mathbb{Z}$-module of rank 8 equipped with the quadratic form (on the canonical basis)

$$q_8(x_1,\ldots,x_8) = 3.\Sigma_{1\leq i\leq 8}\, x_i^2 - 2x_1x_3 - 2x_2x_4 - 3.\Sigma_{3\leq i\leq 7}\, x_i x_{i+1}$$

and H is $\mathbb{Z}^2$ equipped with the hyperbolic quadratic form (on the canonical basis)

$$q_h = 3.x_1x_2$$

Denote the quadratic form on L (or cup-product on $H^2(B,Z)$) by (,). Let $L_{\mathbb{C}} = L \otimes \mathbb{C}$ and for $x \in L_{\mathbb{C}}$ let $[x]$ be its image in $\mathbb{P}L_{\mathbb{C}}$. If

$$\Omega = \{[\omega] \in \mathbb{P}L_{\mathbb{C}} \; ; (\omega,\omega) = 0 \text{ and } (\omega,\overline{\omega}) > 0 \}$$

one can define the period map which to each K3 surface B with a fixed isomorphism

$H^2(B,\mathbb{Z}) \cong L$ associates the line in $L_{\mathbb{C}}$ spanned by the class of a nonzero (nowhere vanishing) differential on B. We denote the period of B by $[\omega_B] \in \mathbb{P}L_{\mathbb{C}}$. Then by [BPV] the image of the period map is Ω.

Let $\{e_i ; 1 \le i \le 8\}$, $\{e_i ; 9 \le i \le 16\}$, $\{f_1, f_2\}$, $\{f_3, f_4\}$, $\{f_5, f_6\}$ be the respective canonical basis for the first and second $-E_8$ summands of L and the three H summands of L. Consider the sublattice L' of L generated by $e_1, e_2, e_5, e_7, e_9, e_{10}, f_1+2f_2$. Then the symmetric bilinear form on L has the same values on our distinguished set of generators of L' as the intersection pairing on H, E_i (with H corresponding to f_1+2f_2). Let $L'^{\perp}$ be the orthogonal complement of L' with respect to the bilinear form on L. Notice that as L' is a primitive sublattice of L (i.e., L/L' has no torsion) we have $L'^{\perp\perp} = L'$.

By [Sh] Chapter IX (p. 216), if B is a K3 surface corresponding to a generic element of Ω orthogonal to L', then the algebraic part of $H^2(B,\mathbb{Z})$ is equal to $L'^{\perp\perp} = L'$.

Given a generic K3 surface $\tilde{B}$ whose period is orthogonal to L', map $\tilde{B}$ to $\mathbb{P}^2$ using the linear system $|H-E_1|$: this linear system is easily seen to be of degree 2 and dimension 2. This map blows E_i down to a point for $i \neq 1$. So if we denote by B' the surface obtained from $\tilde{B}$ by blowing E_i down to points for $i \neq 1$, B' is a double cover of $\mathbb{P}^2$ branched along a plane sextic with five nodes. The image of E_1 in $\mathbb{P}^2$ is an everywhere tangent conic to this sextic. For dimension reasons we see that B' has exactly five ordinary double points as singularities.

Given the plane sextic X $(= X_1)$ with five ordinary double points and everywhere tangent conic $C_t = C_{t1}$, one recovers the quartic surface B :

Let p' be the projection $B' \to \mathbb{P}^2$. The inverse image of C_t in B' has two components C' and C''. By the following, B is the image of B' in $\mathbb{P}^3$ by the linear system $l = |p'^*h+C'|$ where h is the hyperplane class in $\mathbb{P}^2$. (l only collapses C' to a double point of B , and B' has five double points hence B has six double points)

a) The degree of l is 4 . Indeed,

$$(p'^*h+C')^2 = 2h^2 + C'^2 + 2h.C_t = 6 + C'^2$$

and
$$8 = (C'+C'')^2 = 2C'^2 + 2C'.C'' = 2C'^2 + 12 .$$

b) The dimension of l is 3 .

The inverse image of a generic line in $\mathbb{P}^2$ is a curve N of genus 2 . As $p'^*h.C' = 2$, $N \cup C'$ has arithmetic genus 3 . The claim follows by the Riemann-Roch formula, the genus formula and the cohomology sequence associated to the exact sequence

$$0 \longrightarrow \mathcal{O}_B \longrightarrow \mathcal{O}_B(N) \longrightarrow \mathcal{O}_N(N) \longrightarrow 0 .$$

2.3 - The Fano surface E_Z is an intersection of translates of Θ

This is due to Beauville [C3] .

Let X be a curve in $P^{-1}(A)$. Parametrize A and Θ with X (see [M2] and [M3]), take

$$A = \{D : \pi_* D \equiv K_X , h^0(D) \text{ even}\} \subset \text{Pic}^8(\tilde{X})$$
$$\Theta = \{D : h^0(D) > 0\} \subset A$$

Let Z be a double solid such that $X = X_1$ for Z (see 1.9). Let $g^2_6 = |K_X - p - q|$ be the linear system on X associated to the plane representation of X given by Z . Consider a line l in $\tilde{Z}$. Then, looking at the image of l in $\mathbb{P}^2$, one sees that l picks a divisor D_l in g^2_6 which is the sum of the lines through p_1 in Z which are incident to l .

Let p', p'', q', q'' be the liftings of p, q in $\tilde{X}$. By [B3] there is a choice of these liftings, say p', q', such that $h^0(X,D_l+p'+q')$ = dimension of $H^0(X,D_l+p'+q')$ is even as well as $h^0(X,D_l+p''+q'')$. So we obtain an embedding $E_Z \longrightarrow \Theta.\Theta_{p''+q''-p'-q'} = \Theta.\Theta_{[p'',q'']}$

$$l \longmapsto D_l+p'+q'$$

By [C1] the homology class of E_Z is Θ^2, so we have equality.

2.4 - Embeddings of Prym curves in Θ

We prove a fact from [C3] :

Let (X,η) be an element of $\mathcal{P}_5$ such that X is smooth and nonhyperelliptic. Let $A = P(X,\eta)$. Let Y be a curve in $P^{-1}(A)$, tetragonally related to X . As in 2.3 parametrize A and Θ with Y:

$$A = \{D : \pi_* D \equiv K_Y , h^0(D) \text{ even}\} \subset \operatorname{Pic}^8(\tilde{Y})$$

$$\Theta = \{D : h^0(D) > 0\} \subset A$$

Let $g^1_4 \in W^1_4(Y)$ relate Y to X, then $\tilde{X}$ is one component of

$$\{D \in \tilde{Y}^{(4)} : \pi_* D \equiv g^1_4\}$$

THEOREM : *Each Prym-embedding of* $\tilde{X}$ *in* Θ *is given by*

$$\tilde{X}_E = \{D+E : D \in \tilde{X}\}$$

for $E \in \tilde{X}'$ *where* $\tilde{X}'$ *is one of the two components of the set*

$$\{E \in \tilde{Y}^{(4)} : \pi_* E \equiv K_Y - g^1_4\}$$

Proof: Parametrize A and Θ with X. Then a copy of $\tilde{X}$ in A is of the form $\tilde{X}_E = \{p - \sigma p + E : p \in \tilde{X}\}$ for some divisor E on $\tilde{X}$ of degree 8, such that $\pi_* E \equiv K_X$ and $h^0(E)$ is odd. Therefore by Clifford's theorem $h^0(E) = 1$ or 3 because X is nonhyperelliptic. As

$$\tilde{X}_E \cap \Theta = \{p : h^0(p - \sigma p + E) > 0\}$$

it is immediately seen that $\tilde{X}_E$ is in Θ if and only if $h^0(E) = 3$. So the set of Prym-embeddings of $\tilde{X}$ in Θ is

$$\tilde{X}'' = \{E \in \operatorname{Pic}^8 \tilde{X} : \pi_* E \equiv K_X \text{ and } h^0(E) = 3\} .$$

Notice that $\tilde{X}' \subset \tilde{X}''$. By [B3] the homology class of $\tilde{X}'$ is $[\Theta]^3/3$. By [W2] (see also [B]) $\tilde{X}''$ is a smooth curve for A generic and its homology class is also $[\Theta]^3/3$. Hence they are equal if A is generic.

In general, choose a generic point $p \in \tilde{X}$. Then (see [T]) the map $E \longmapsto E+p-\sigma p$ defines an isomorphism of $\tilde{X}''$ with

$$W_p = \{D : h^o(D-p) > 0\} \subset \Theta .$$

As p is generic we see that the proof of proposition 1 and the remark following it in [BD1] apply : W_p is a curve with homology class $[\Theta]^3/3$.

Hence $\tilde{X}' = \tilde{X}''$.

2.5 - Embeddings of Prym curves in intersections of two translates of Θ

We write down the result of [De] :

Consider a tetragonally related triple of smooth, nonhyperelliptic and nontrigonal curves X, Y, U in $P^{-1}(A)$. Let $g_4^1 \in W_4^1(Y)$ relate Y to X and U. If Y is bielliptic, suppose that g_4^1 is not the pullback of a g_2^1 on an elliptic curve. Then for each [p,q] and [p,r] in $\Sigma(Y)$ such that p and q are not images of each other by a bielliptic involution of Y and such that there exists $x \in \tilde{Y}$ with $\pi p+\pi q+\pi r+\pi x \equiv g_4^1$:

PROPOSITION : *The pairs* [p,q] *and* [p,r] *are also elements of* $\Sigma(X)$ *and* $\Sigma(Z)$. *The intersection*

$$\Theta.\Theta_{[p,q]}.\Theta_{[p,r]} = S_{pqr} \cup W_p$$

is the union of translates of $\tilde{X}', \tilde{Y}', \tilde{U}'$ *with the notations of* 2.4 .

Proof : By [BD1] we have :

$$\Theta.\Theta_{[p,q]}.\Theta_{[p,r]} = S_{pqr} \cup W_p$$

Where S_{pqr} is defined to be

$$S_{pqr} = \{D ; h^0(D-p-q-r) > 0\} \subset \Theta .$$

The homology classes of W_p and S_{pqr} are respectively $[\Theta]^3/3$ and $2[\Theta]^3/3$ by [BD1] and [B3].

We have

$$|K_Y - \pi p - \pi q - \pi r| = \pi x + |K_Y - g_4^1|$$

as linear systems and by the definition of the tetragonal construction in 1.3, S_{pqr} splits as the union of

$$p+q+r+x+\tilde{X}' \quad \text{and} \quad p+q+r+\sigma x+\tilde{U}'$$

where X', U' are the two curves tetragonally related to Y via $K_Y - g_4^1$. Now, using 3.2 (we do not use 2.5 to prove 3.2), it is easy to see that $[p,q]$ and $[p,r]$ are also elements of $\Sigma(X)$ and $\Sigma(Z)$.

2.6 - Prym-embeddings of $\tilde{X}_{ij}$ in E_Z

This is due to Clemens.

PROPOSITION : *For each* i, j *there are (two) Prym-embeddings of* $\tilde{X}_{ij}$ *in* E_Z.

Proof : As we noted in the introduction, for all $i \neq j$, X_i, X_j and X_{ij} are tetragonally related. The g_4^1 on X_i is the one given by lines through the image of p_j in the plane representation of X_i. Take $i = 1$, $j = 2$. Let l_t be a line through p_2, corresponding to a point $t \in X_2$. Then l_t projects to a line through the image of p_2 in $\mathbb{P}^2$ and picks (via incidence) a lifting, say $s_1+s_2+s_3+s$ of a divisor in $|g_6^2 - p_2' - p_2''|$, where p_2' and p_2'' are the two points of X_1 above p_2. The pair of incident lines (l_s,l_t) through p_1 and p_2 corresponding to $(s,t) \in X_{12} \subset X_1 \times X_2$ picks, via incidence, the divisor $s_1+s_2+s_3+\sigma s$ in $|g_6^2 - p_2' - p_2''|$.

Recall that an "actual" line in $\tilde{Z}$ is, for instance, the union of l_s and a ruling of $\mathcal{Q}_1$ (this is analogous to the case of conics, see 2.1). However, we will not need to take this into account in our computations.

Let V be the plane spanned by $\tilde{\pi}(l_s)$ and $\tilde{\pi}(l_t)$. Then $\tilde{V} = \tilde{\pi}^{-1}(V)$ is a Del Pezzo surface isomorphic to $\mathbb{P}^2$ blown up at 7 points q_i $(0 \leq i \leq 6)$ such that q_0, q_1, q_2 and q_0, q_3, q_4 are colinear. The plane V is the image of $\tilde{V}$ by the map defined by the linear system of strict transforms of cubics in $\mathbb{P}^2$ passing through the q_i's. The lines $\langle q_0,q_1\rangle$

and $\langle q_0,q_3\rangle$ are blown down to p_1 and p_2. By [Dm] a basis of the Picard group of $\tilde{V}$ is given by

$$H\,,\,-E_0\,,\,\dots\,,\,-E_6$$

where H is the pullback of the hyperplane class in $\mathbb{P}^2$ and E_i is the exceptional divisor above q_i.

When the q_i's are in general position, the lines in $\tilde{V}$ are given by :

- exceptional curves E_i (7 curves)

and the strict transforms of the following curves in $\mathbb{P}^2$

- lines through 2 of the q_i's (21 curves)
- conics through 5 of the q_i's (21 curves)
- cubics through all the q_i's and with a double point at one of them (7 curves).

In our degenerate case, some of the above curves coincide.

If, for instance, $l_s = L_{35}$ (L_{ij} is the strict transform of the line through q_i and q_j) , and $l_t = E_3$, we have

$$l_s + l_t = H + (-E_5) = L_{56} + E_6 = E_0 + L_{05}$$

Notice that E_0 and L_{56} are the two lines in $\tilde{V}$ which project to $\langle p_1,p_2\rangle$ in $\mathbb{P}^3$. So

$$l_{st} = l_s + l_t - L_{56} = E_6 \quad \text{and} \quad (l_{st})' = l_s + l_t - E_0 = L_{05}$$

are lines in $\tilde{Z}$. Similarly, the same holds for the other pairs of incident lines. Thus we obtain two Prym-embeddings of $\tilde{X}_{12}$ in E_Z ($(l_s,l_t) \mapsto l_{st}$ and $(l_s,l_t) \mapsto (l_{st})'$).

3. PRELIMINARIES

We prove some preliminary results that we will need later on.

Throughout this section, unless otherwise stated, A is a generic element of $\mathcal{A}_4^0$.

(3.1) Let $(\tilde{X},X)$ and $(\tilde{Y},Y)$ be two smooth elements of $P^{-1}(A)$. Define $\tilde{X}_\lambda$ to be the curve parametrizing Prym-embeddings of $\tilde{X}$ in Θ. By 2.4, $\tilde{X}_\lambda$ has a fixed point free involution σ_λ. Let X_λ be the quotient of $\tilde{X}_\lambda$ by σ_λ. By 2.4, if Y is tetragonally related to X through g_4^1 on Y, then Y is tetragonally related to X_λ through $K_Y - g_4^1$. Hence X_λ is of genus 5 and $P(\tilde{X}_\lambda,X_\lambda) = A$. Hence if we choose a double solid (as in the introduction) such that, with the notations of the introduction, $X = X_1$, then $X_\lambda = X'$ (take $Y = X_2$). *From this it becomes clear that* $\lambda : (\tilde{X},X) \mapsto (\tilde{X}_\lambda,X_\lambda)$ *is an involution. Also* $(\tilde{X}_\lambda,X_\lambda) \neq (\tilde{X},X)$ *in general*. This can be seen for instance by doing the same construction for A = JC the jacobian of a smooth, irreducible curve (see 4.2, notice that we do not use this in 4.2).

(3.2) **PROPOSITION** : *The intersection* $\Theta.\Theta_x$ *contains a Prym-translate of* $\tilde{X}_\lambda$ *if and only if* $x \in \Sigma(X)$. *In that case,* $\Theta.\Theta_x$ *contains exactly two copies of* $\tilde{X}_\lambda$ *which correspond to* s, t *where* $x = [s,t]$.

Proof : Looking back at 2.4 we see that we have a surface in Θ traced by the Prym-embeddings of $\tilde{X}$ (or $\tilde{X}_\lambda$) in Θ. Considering A naturally embedded in $J\tilde{X}$ (resp. $J\tilde{X}_\lambda$) these embeddings are translates of

$$\{p - \sigma p \,;\, p \in \tilde{X}\}$$

$$(\text{resp.} \quad \{p - \sigma p \,;\, p \in \tilde{X}_\lambda\}\,)$$

in $J\tilde{X}$ (resp. $J\tilde{X}_\lambda$). Abusing notations, we denote by $p - \sigma p$ an element of A which is the translate of $p - \sigma p \in J\tilde{X}$ (resp. $J\tilde{X}_\lambda$) by a *fixed* element of $J\tilde{X}$ (resp. $J\tilde{X}_\lambda$).

So the above surface is the set of elements $a+p-\sigma p+q-\sigma q$ for $p \in \tilde{X}, q \in \tilde{X}_\lambda$ and some fixed $a \in A$.

Suppose that we have an embedding of $\tilde{X}_\lambda$ in $\Theta.\Theta_x$ for $x \in A$.

Then there exists $p \in \tilde{X}$ such that

$$a+p-\sigma p+q-\sigma q \in \Theta.\Theta_x \text{ for all } q \in \tilde{X}_\lambda$$

that is

$$a+p-\sigma p+q-\sigma q - x \in \Theta \text{ for all } q \in \tilde{X}_\lambda$$

This implies that there exists $p' \in \tilde{X}$ such that

$$a+p-\sigma p+q-\sigma q - x = a+p'-\sigma p'+q-\sigma q \quad \text{for all } q \in \tilde{X}_\lambda$$

so we obtain $x = p-\sigma p-p'+\sigma p' = [p,\sigma p'] \in \Sigma(X)$ and in fact we have exactly 2 embeddings of $\tilde{X}_\lambda$ in $\Theta.\Theta_x$.

Conversely, for all $x \in \Sigma(X)$ and $q \in \tilde{X}_\lambda$ the Prym-translate of $\tilde{X}$ in Θ corresponding to q and its translate by x intersect in two points which trace two copies of $\tilde{X}_\lambda$ in $\Theta.\Theta_x$ as q varies.

Notice that, with the notations of 2.4 and 2.5, the Prym-translates of $\tilde{X}_\lambda$ in $\Theta.\Theta_{[s,t]}$ for $[s,t] \in \Sigma(X)$ are just W_s and W_t. Q.E.D.

(3.3) **LEMMA** : *Suppose that* Z *is a double solid such that* $X = X_1$ *is obtained by projection from one of its double points* p_1 *(cf 2.2), then* $\Sigma(X) \subset D_Z$.

Proof : We first notice that $\Sigma(X)$ is equal to the image by AJ (with base point the inverse image in $\tilde{Z}$ of a line in $\mathbb{P}^3$ tangent to B) of

$$\{l_p \cup L^1_p \cup L^2_q \cup l_q \,;\, p, q \in \tilde{X}\}$$

where l_p, l_q are the strict transforms in $\tilde{Z}$ of the lines in Z corresponding to p, q ; L^1, L^2 are the two rulings of the exceptional quadric $\mathcal{Q}_1$ in $\tilde{Z}$ above p_1 and L^1_p is the line of the ruling L^1 which passes through $\mathcal{Q}_1 \cap l_p$. By definition, a conic in $\tilde{Z}$ is a curve C verifying

- $C.\tilde{\pi}^*\mathcal{O}(1) = 2$

$- C.\mathcal{Q}_i = 0$ for all i .

The restriction of the ideal sheaf of $\mathcal{Q}_1$ to $\mathcal{Q}_1$ is $\mathcal{O}_{\mathcal{Q}_1}(-1,-1)$. So to obtain an actual conic in $\tilde{Z}$ with respect to $\tilde{\pi}^*\mathcal{O}(1)$ we have to add the two rulings of $\mathcal{Q}_1$ to $l_p \cup l_q$. Also

$$AJ(l_p \cup L_p^1 \cup L_q^2 \cup l_q) = AJ(l_p \cup L_p^2 \cup L_q^1 \cup l_q) .$$

This is because both surfaces are equal to the union of the Prym-embeddings of $\tilde{X}$ which pass through the origin.

Let V be the plane spanned by $\tilde{\pi}(l_p)$, $\tilde{\pi}(l_q)$ in $\mathbb{P}^3$ and define $L_{pq} = \mathcal{Q}_1 \cap \tilde{\pi}^{-1}(V)$. Then L_{pq} is a (1,1)-curve in $\mathcal{Q}_1$. Hence

$$AJ(l_p \cup L_p^1 \cup L_q^2 \cup l_q) \;=\; AJ(l_p \cup L_{pq} \cup l_q) .$$

We claim that $l_p \cup L_{pq} \cup l_q$ is in the linear equivalence class of a smooth rational conic in $\tilde{\pi}^{-1}(V)$ except when $p = \sigma q$. In the latter case, the image of $l_p \cup L_{pq} \cup l_{\sigma p}$ is twice a line and

$$AJ(l_p \cup L_{pq} \cup l_q) \;=\; 0$$

with the convention of 2.1 .

Indeed, as in 2.6, $\tilde{V} = \tilde{\pi}^{-1}(V)$ is a Del Pezzo surface isomorphic to the blow up of $\mathbb{P}^2$ at 7 points q_i . The 7 points are not in general position but, as $\pi^{-1}(V)$ has a double point, 3 of them, say q_1, q_2, q_3 , lie on a line which is precisely L_{pq} . The arithmetic genus of $C = l_p \cup L_{pq} \cup l_{\sigma p}$ is 0 . The map $\tilde{V} \to V \cong \mathbb{P}^2$ is given by the anticanonical bundle $-K_{\tilde{V}}$ of $\tilde{V}$ [Dm] . Hence, as the image of C is a conic, we have

$$-K_{\tilde{V}} . C \;=\; 2 .$$

The genus formula then gives $C.C = 0$. Using Riemann-Roch and the cohomology sequence associated to the exact sequence

$$0 \;\to\; \mathcal{O}_{\tilde{V}} \;\to\; \mathcal{O}_{\tilde{V}}(C) \;\to\; \mathcal{O}_C(C) \;\to\; 0$$

one sees that C moves in a linear system of (projective) dimension 1 . The arithmetic genus of C being 0 , the generic member of this pencil is a smooth rational conic.Q.E.D.

(3.4) **LEMMA** : *With the notations of* 1.9, *for a generic double solid* Z *and any discriminant curve* X_{ij} *for* Z :

$$\Sigma(X_{ij}) \not\subset D_Z$$

Proof : As in 2.6, let (l_u,l_v) represent an element of $\tilde{X}_{12}$. Then, with the notations of the proof of 3.3,

$$\{AJ(L_u^i \cup l_u \cup l_v \cup L_v^j) \,;\, (l_u,l_v) \in \tilde{X}_{12}\}$$

is a Prym-embedding of $\tilde{X}_{12}$ in D_Z and hence in A , for all $i, j \in \{1,2\}$. As before the base point for the Abel-Jacobi mapping is the inverse image in $\tilde{Z}$ of a line in $\mathbb{P}^3$ tangent to B . Thus, still with the notations of the proof of 3.3, a translate of

$$\{ AJ\,(l_s \cup l_t \cup l_u \cup l_v \cup L_{su} \cup L_{tv}) \,;\, (l_s,l_t), (l_u,l_v) \in \tilde{X}_{12} \}$$

is equal to $\Sigma(X_{12})$. Actually, with our choice of base point, this is exactly $\Sigma(X_{12})$ because it is the union of the Prym-embeddings of $\tilde{X}_{12}$ that pass through the origin.

The image of $l_s \cup l_t \cup l_u \cup l_v \cup L_{su} \cup L_{tv}$ by the Abel-Jacobi mapping is equal to the rational equivalence class of

$$l_s + l_t + l_u + l_v + L_{su} + L_{tv} - l_s - l_{is} - L_s^1 - L_s^2 - l_t - l_{it} - L_t^1 - L_t^2$$

or, as $L_{su} - L_s^1 - L_s^2$ and $L_{tv} - L_t^1 - L_t^2$ are rationally equivalent to 0 , to

$$l_u + l_v - l_{is} - l_{it} \;=\; l_u + l_v - il_s - il_t$$

or, by 2.6, to

$$l_{uv} - il_{st} \;=\; l_{uv} + l_{st} - l_{st} - il_{st}$$

For (u,v) and (s,t) generic the lines l_{uv} and l_{st} do not intersect each other so $AJ(l_{uv} + l_{st})$ is not an element of D_Z . Q.E.D.

Recall that in 1.9 we introduced the (Cremona) group $\mathcal{G}$ of birational transformations of $\mathbb{P}^3$ and described its orbits in $\mathcal{Z}_A$ (see notations). We have the following :

(3.5) **PROPOSITION** : *Let* $\{Z_1, \iota Z_1,\ldots, Z_{16}, \iota Z_{16}\}$ *be the orbit under* $\mathcal{G}$ *of a generic element* Z_1 *of* $\mathcal{Z}_A$ *. Then*

i) $E_{Z_j} = E_{\iota Z_j}$ *for all* j *and*

ii) $E_{Z_j} \neq E_{Z_k}$ *for all* $j \neq k$.

(By equality (resp. inequality) we mean that they are (resp. are not) images of each other by an automorphism of A.)

Proof: The map $\rho : \mathbb{P}^3 \longrightarrow \mathbb{P}^3$ given by the linear system of quadrics through the nodes p_i of Z is the composition :

$$\mathbb{P}^3 \overset{\mathcal{O}(2)}{\hookrightarrow} \mathbb{P}^9 \dashrightarrow \mathbb{P}^3$$

where the arrow on the right is the projection from the nodes of Z . ρ is a rational map of degree 2 and blows up the double points of Z . So we can define the rational involution ι which interchanges the sheets of ρ .

The rational involutions $\iota_{\mathfrak{F}}$ are compositions of

$$\mathbb{P}^3 \overset{\mathcal{O}(3)}{\hookrightarrow} \mathbb{P}^{19} \dashrightarrow \mathbb{P}^3$$

where the arrow on the right is projection from the projective tangent spaces to the image of $\mathbb{P}^3$ in $\mathbb{P}^{19}$ at the points of $\mathfrak{F}$ where $\mathfrak{F}$ is a subset of cardinality 4 of $\{p_1,\dots, p_6\}$.

i) For a line l in $\mathbb{P}^3$ (bitangent to ιB but otherwise generic) $\iota(l)$ is a curve of degree 7 with double points at the p_i's and elsewhere tangent to B .

So we need to show that for each line l' in $\tilde{Z}$ there is a curve C' in $\tilde{Z}$ (isomorphic to its projection C in $\mathbb{P}^3$, such that C is a septic everywhere tangent to B and passes doubly through the p_i's) such that $l' \cup C'$ is a complete intersection in $\tilde{Z}$. (Then the image of $l' \cup C'$ under AJ is constant when l' varies.)

Let l be $\pi(l')$, q the unique quadric in $\mathbb{P}^3$ containing l and the p_i's . Let $q' \neq q$ be a quadric from the pencil of those passing through the p_i's and the two points of contact of l with B .

In the pencil spanned by B and $(q')^2$ there exists a quartic B' containing l . Let B, B', q' denote also the equations of B, B' and q' . Write $B' = B - r(q')^2$ for a complex number r. Then, as $B = N^2$ in $\tilde{Z}$, the equation of B' in $\tilde{Z}$ is $(N - \sqrt{r}.q').(N + \sqrt{r}.q')$. Hence the inverse image of B' in $\tilde{Z}$ is the union of two surfaces one of which, say Q, contains l'. Then $Q \cap \tilde{\pi}^{-1}(q)$ is a complete intersection of the required type. (Similar proofs are in [C3] and [W].)

ii) Consider a subset $\mathfrak{F} \subset \{p_1,\ldots, p_6\}$ of cardinality 4, say $\mathfrak{F} = \{p_1,\ldots, p_4\}$ and let $Z' = \iota_{\mathfrak{F}}Z$. The plane spanned by p_2, p_3, p_4 is blown down by $\iota_{\mathfrak{F}}$ to a double point p'_1 of Z' and lines through p_1 in Z go to lines through p'_1 in Z'. So if X is the discriminant curve for the projection of B from p_1, then X is also the discriminant curve for the projection of $B' = \iota_{\mathfrak{F}}B$ from p'_1.

Projecting from p_1 and p'_1 we have two plane representations of X as a sextic with five double points that are not projectively equivalent because they determine the nonprojectively equivalent quartics B and B' (2.2). Suppose that these plane representations are given by the linear systems $|K - p - q|$, $|K - p' - q'|$. By 2.3, for some choice of liftings of p, q, p', q' say p_1, q_1, p'_1, q'_1 in $\tilde{X}$, we have :

$$E_Z = \Theta.\Theta_{[p_1,q_1]} \ , \ E_{Z'} = \Theta.\Theta_{[p'_1,q'_1]}$$

as $\{p,q\} \neq \{p',q'\}$ these two varieties cannot be images of each other by an automorphism of A. Q.E.D.

Let ω_X be the invertible sheaf of regular differentials on X and Ω_A be the rank 4 free sheaf of regular differentials on A. Let $S^4 = S^4H^0(A,\Omega_A) = S^4H^0(X,\omega_X \otimes \eta)$ be the space of symmetric quadrilinear forms on T_0A. We have a linear map

$$\tau : \Gamma_{00} \to S^4$$

which to each element of Γ_{00} associates the quartic term of its Taylor expansion at 0.

(3.6) **PROPOSITION** : *For* $A = JC$ *the jacobian of a nonhyperelliptic curve* C *of genus* 4 *with a unique* g^1_3 *(denoted by* ξ *), the mapping* τ *is injective. (* A *is an element of* $\theta_{null} \cap \mathfrak{J}_4$ *)*

Proof : For $\zeta \in \mathrm{Pic}^3C$, let $\Theta_\zeta = W_3 - \zeta$. Let θ_ζ be a nonzero section of $\mathcal{O}_A(\Theta_\zeta)$.

For $D \in T_0A$, denote by $D\theta_\zeta$ be the derivative of θ_ζ with respect to ζ, in the direction D. Let

$$\theta_\xi = q + g + \ldots$$

be the Taylor expansion of θ_ξ at 0, with q and g irreducible polynomials of respective degrees 2 and 4 ([ACGH]). The quadric

$$q \in S^2H^0(A,\Omega_A) = S^2H^0(C,\omega_C) = \mathrm{Hom}(H^0(A,\Omega_A)^*,H^0(A,\Omega_A)) = \mathrm{Hom}(T_0A,(T_0A)^*)$$

has rank 3. Let $D_0 \in T_0A$ be a generator of its kernel. Then ([BD2]) the canonical model of C is the complete intersection of $\overline{q} = \{q = 0\}$ and $\overline{f} = \{f = D_0g = 0\}$, the space Γ_{00} is generated by $(\theta_\xi)^2$ and the $DD_0\theta_\xi.\theta_\xi - D_0\theta_\xi.D\theta_\xi$ for $D \in T_0A$. The quartic terms of the Taylor expansions at 0 of $(\theta_\xi)^2$ and the $DD_0\theta_\xi.\theta_\xi - D_0\theta_\xi.D\theta_\xi$ are respectively

$$q^2 \quad \text{and} \quad 2(q.Df - f.Dq)$$

If τ is not injective, then there exist $D \in T_0A$ and $r \in \mathbb{C}$ with $(r,D) \neq (0,0)$ such that

$$r.q^2 + q.Df - f.Dq = 0$$

i.e.,

$$q.(r.q + Df) = f.Dq$$

As q and f are irreducible this implies

$$r.q + Df = Dq = 0$$

Hence D is a multiple of D_0. We can suppose $D = D_0$. We get

$$r.q + D_0f = 0$$

and applying D_0 to this equation we obtain $(D_0)^2f = 0$. This would imply that the image of D_0 in $\mathbb{P}T_0A$ is in $\overline{q} \cap \overline{f} = C$. The ruling of $\overline{q}$ cuts on C the divisors of ξ. If the singular point of $\overline{q}$ is on C, then the ruling of $\overline{q}$ will cut a g^1_2 on C and κC will have a double point at the vertex of $\overline{q}$: both of these are impossible. Q.E.D.

(3.7) *Remark* : This result can be proved in a similar way for a curve C with two distinct

g_3^1's . It can also be proved for a hyperelliptic curve in a different way. However, we do not need these here.

4. INTERSECTIONS OF TWO TRANSLATES OF Θ

In this section our aim is to compute the number of Prym-embedded curves in an intersection of two translates of theta .

(4.1) **PROPOSITION** : *For* $A = JC$ *the jacobian of a generic curve* C *of genus* 4 *, a generic intersection of two translates of* Θ *contains translates of* 27 *distinct Prym-embedded curves.*

Proof :

(4.2) The fiber of the Prym map at A splits into two components each isomorphic to $C^{(2)}$ (the second symmetric product of C) (see [B1]) and exchanged by the involution λ :

One component is the space whose generic elements are trigonal curves related to C via Recillas' construction ([R]). More precisely, given a $g^1_4 = K_C - p - q$ on C , one defines $\tilde{X}$ as the set of elements $\{s,t\}$ of $C^{(2)}$ such that $h^0(g^1_4 - s - t) > 0$. The involution σ on $\tilde{X}$ is defined by sending $\{s,t\}$ to $\{s',t'\}$ whenever $s+t+s'+t' \in g^1_4$. The points of a divisor of the g^1_3 on $X = \tilde{X}_{/\sigma}$ are the three partitions of a divisor of g^1_4 into two sets of two points.

The other component is the space of singular curves with one double point whose normalization is C . If C_{pq} is the curve obtained from C by identifying the two points p and q , the double cover $\tilde{C}_{pq}$ of C_{pq} with A as Prym variety is obtained by taking two copies of C and identifying p on one copy with q on the other.

(4.3) View A as Pic^3C and take $\Theta = W_3$ (the effective divisor classes). Then for each trigonal X we see immediately that we have the following embeddings of $\tilde{X}$ in Θ (and only these) :

$\{s,t\} \mapsto s+t+p_0$

or by $\{s,t\} \mapsto K_C - (s+t+p_0)$ for all $p_0 \in C$.

As the family of the above embeddings is $\tilde{C}_{pq}$, we deduce that $X_\lambda = C_{pq}$.

Analogously, the embeddings of $\tilde{C}_{pq} = C_1 \cup_{p=q} C_2$ ($C_i \cong C$) are given by :

$C_1 \ni s \mapsto s_1+s_2+s$

$C_2 \ni t \mapsto K_C - t - t_1 - t_2$ for all s_1,s_2,t_1,t_2 such that $s_1+s_2+t_1+t_2 \equiv g^1_4 \equiv K_C - p - q$.

(4.4) We have a surjective morphism of *generic* degree 6 :

$C^{(2)} \times C^{(2)} \longrightarrow A$

$(a+b, a'+b') \mapsto a+b - a'-b'$

If (generically) $a+b - a'-b' = c+d - c'-d'$ for distinct elements of $C^{(2)} \times C^{(2)}$, then

$a+b+c'+d' \equiv c+d+a'+b' \equiv K_C - p - q$ for some $p,q \in C$.

So the number of pairs $(c+d, c'+d')$ with this property (for fixed a,b,a',b') is equal to the number of bisecant lines $\langle p,q \rangle$ to the canonical model of C which intersect simultaneously the lines $\langle a,b \rangle$ and $\langle a',b' \rangle$. Equivalently, we need to have $h^0(g - p - q) > 0$ and $h^0(g' - p - q) > 0$ where $g = K_C - a - b$ and $g' = K_C - a' - b'$.

The preceding inequalities define two correspondences on $C \times C$ and the number we are looking for is half the intersection number of these correspondences. By [GH] (p. 285) these correspondences are homologous to $4E + 4F - \Delta$ where E and F are fibers of the two projections $C \times C \longrightarrow C$ and Δ is the diagonal. The intersection number is $(4E + 4F - \Delta)^2 = 33.E.F - 8E.\Delta - 8F.\Delta + \Delta^2 = 10$ (as $\Delta^2 = -(2g-2) = -6$), half of it is 5 and the degree is $5+1 = 6$.

(4.5) We will now compute the number of Prym-embedded curves in $\Theta.\Theta_x$ for x generic in A. We will first consider trigonal curves. Our notations will be as in 4.2 and 4.3.

The set of effective divisors in $\mathrm{Pic}^1 C$ is one-dimensional (= C) and, for x generic, C+x will not intersect it. Hence we can assume $h^0(p_0+x) = 0$ for all $p_0 \in C$. Equivalently $h^0(K_C - p_0 - x) = 2$. Let $g_5^1 = K_C - p_0 - x$. By 4.3, we need to know when for all $\{s,t\} \in \tilde{X}$ we will have

1) $h^0(s+t+p_0+x) > 0$ or

2) $h^0(K_C - s - t - p_0 + x) > 0$.

By Riemann-Roch, case 2) is equivalent to case 1) with x replaced by -x . By Riemann-Roch again, case 1) is equivalent to :

For all s,t , if $h^0(g_4^1 - s - t) > 0$, then $h^0(g_5^1 - s - t) > 0$.

This is possible if and only if $g_5^1 = g_4^1 + t_0$ for some t_0 in C , or if and only if $t_0+p_0+x = K_C - g_4^1 = q_0+s_0$ is an effective divisor.

Taking into consideration the cases 1 and 2, we see that for each of the 6 representatives $a_i+b_i-a_i'-b_i'$ $(1 \le i \le 6)$ of x in $C^{(2)} \times C^{(2)}$ we have (two) embeddings of the trigonal curves obtained from $K_C - a_i - b_i$ and $K_C - a_i' - b_i'$ in $\Theta.\Theta_x$.

Looking at (4.4) we see that for all $i \neq j$ there is a line l_{ij} bisecant to the canonical model of C which encounters simultaneously $\langle a_i,b_i\rangle$, $\langle a_i',b_i'\rangle$, $\langle a_j,b_j\rangle$ and $\langle a_j',b_j'\rangle$. Hence the number of lines incident to $\langle a_i,b_i\rangle$, $\langle a_i',b_i'\rangle$ for some i (or incident to $\langle a_i,b_i\rangle$, $\langle a_j',b_j'\rangle$ for some i , j) is 15 .

By 4.3, in the singular case, $\tilde{C}_{pq}$ embeds in $\Theta.\Theta_x$ if and only if for some s_1,s_2,t_1,t_2 such that $s_1+s_2+t_1+t_2 \equiv g_4^1 \equiv K_C - p - q$:

$h^0(t+t_1+t_2+x) > 0$ and $h^0(K_C - t - s_1 - s_2 + x) > 0$ for all $t \in C$. Equivalently :

$h^0(t_1+t_2+x) > 0$ and $h^0(s_1+s_2-x) > 0$.

So we must have $t_1+t_2 = a_i'+b_i'$, $s_1+s_2 = a_j+b_j$ and $\langle p,q\rangle = l_{ij}$ for some $i \neq j$.

Counting everything we obtain 27 curves in $\Theta.\Theta_x$. Q.E.D.

(4.6) **PROPOSITION** : *A generic element of* A *lies on exactly* 27 *surfaces* $\Sigma(X)$.

Proof : Let $\mathcal{H}$ be the Siegel half space of symmetric 4×4 complex matrices whose imaginary part is positive definite. On $\mathcal{H}$ we have a universal family of abelian varieties $\mathbb{A}$ and a universal theta divisor defined by Riemann's theta function. Let $\mathfrak{C}_A \subset A \times P^{-1}(A)$ be the reduced variety of all pairs (a,X) such that $a \in \Sigma(X)$. Pulling back we have a diagram :

$$\begin{array}{ccccccccc} \cup_A \mathfrak{C}_A = \mathfrak{C} & \subset & \mathfrak{F} & \to & \mathfrak{R} & \to & \mathfrak{P}_5 \\ p\searrow & & P\downarrow & & P\downarrow & & P\downarrow \\ & & \mathbb{A} & \to & \mathcal{H} & \to & \mathfrak{A}_4 \end{array}$$

The number we are interested in is the generic degree of p. Using lemma 4.1 we need to show that p is unramified at a generic jacobian.

The cotangent space to $\mathfrak{P}_5$ at (X,η) is canonically isomorphic to $H^0(X,\omega_X^2)$ [B1] . We have the following exact sequences of cotangent spaces at $(a,X) \in \mathfrak{C}$:

$$0 \to H^0(X,\omega_X^2) \to T^*_{(a,X)}\, \mathfrak{C} \to T^*_a\, \Sigma(X) \to 0$$

$$0 \to S^2H^0(X,\omega_X \otimes \eta) \to T^*_{(a,X)}\, \mathbb{A} \to H^0(X,\omega_X \otimes \eta) \to 0$$

We have a surjection : $H^0(X,\omega_X \otimes \eta) \twoheadrightarrow T^*_a\, \Sigma(X)$

and multiplication : $S^2H^0(X,\omega_X \otimes \eta) \hookrightarrow H^0(X,\omega_X^2)$ (i.e., the Prym map is unramified [B1]).

Hence at a *generic abelian variety* we have an isomorphism between $H^0(X,\omega_X^2)$ and the kernel of the composition $T^*_{(a,X)}\, \mathbb{A} \to H^0(X,\omega_X \otimes \mu_X) \to T^*_a\, \Sigma(X)$. Or an isomorphism between the kernel N of $H^0(X,\omega_X \otimes \mu_X) \to T^*_a\, \Sigma(X)$ and $H^0(X,\omega_X^2) / S^2H^0(X,\omega_X \otimes \mu_X)$. This corresponds to the deformations of X in $\mathbb{P}T_0A$ and means that X moves in two directions transversal to the line $\langle p,q \rangle$ where $a = [p,q] \in \Sigma(X)$. Equivalently the deformations of X (i.e., the curves in $P^{-1}(A)$) fill $\mathbb{P}T_0A$ (because a can take any generic value in $\Sigma(X)$).

These remain valid when $A = JC$ is a generic jacobian. For C_{st} a singular curve, the Prym-canonical embedding of C_{st} is the union of the canonical model of C and the line $\langle s,t \rangle$ ([B2]). A generic quadric in $\mathbb{P}T_0A$ does not contain $\langle s,t \rangle$, hence the Prym mapping is still unramified. The lines $\langle s,t \rangle$ fill $\mathbb{P}^3$, hence the deformations of C_{st} fill $\mathbb{P}^3$. Letting λ act, we obtain the same result for X trigonal in $P^{-1}(JC)$. Q.E.D.

(4.7) **COROLLARY** : *A generic intersection of two translates of* Θ *in a generic abelian variety* A *of dimension* 4 *contains translates of* 27 *different Prym-embedded curves.*

Proof: By (3.2), for $a \in A$ and $X \in P^{-1}(A)$, $\tilde{X}$ admits an embedding as a Prym-curve in $\Theta.\Theta_a$ if and only if $a \in \Sigma(X_\lambda)$. Now use 4.6.

(4.8) **PROPOSITION** : *For a double solid* Z *in* $\mathcal{Z}$ *, the Prym-embedded curves in* E_Z *are the*

$$\tilde{X}_i \ , \ (\tilde{X}_i)_\lambda \ , \ \tilde{X}_{ij} \ .$$

Proof : Clearly, we have an embedding of $\tilde{X}_i$ in E_Z for each choice of ruling of the exceptional quadric above p_i in $\tilde{Z}$. As $E_Z = E_{\iota Z}$, we have two embeddings of $(\tilde{X}_i)_\lambda$ in E_Z. By 2.6, we also have two embeddings of $\tilde{X}_{ij}$. So we know all the 27 curves.

5. THE CUBIC THREEFOLD AND THE $|2\Theta|_{00}$ DIVISORS

We are going to construct the cubic threefold. Throughout this section, A is a generic ppav.

(5.1) To each curve $X \in P^{-1}(A)$ we associate a pencil l_X of Γ_{00}-divisors in the following way :

There is a one-dimensional family $\mathcal{Z}_X$ of double solids Z such that $X = X_1$ is a discriminant curve for the projection from one of the double points p_1 of Z. Recall (3.1) that $X'_1 = (X_1)_\lambda = X_\lambda$. Looking at the action of the birational involution ι, one sees that X_λ is the discriminant curve for the projection of ιZ from p_1 [C3].

Consider the Γ_{00}-divisor D_Z associated to Z (2.1), then by 3.6, since A is generic, the map τ is injective. As the tangent cone at 0 to D_Z is $\rho(B) = \rho(\iota B)$ (see 2.1) we have $D_Z = D_{\iota Z}$. Hence, using 3.3, one sees that for all Z as above, D_Z contains $\Sigma(X) \cup \Sigma(X_\lambda)$.

Using Pontrjagin product and (A) (in 2.1) we see that the homology class of $\Sigma(X)$ is $2\Theta^2$. Hence the set of Γ_{00}-divisors D_Z containing $\Sigma(X) \cup \Sigma(X_\lambda)$ is a pencil that we denote by $l_X = l_{X_\lambda}$ (no other curve Y verifies $l_Y = l_X$).

Notice that for all $i \in \{1,\ldots,16\}$, D_{Z_i} (see 3.3) contains $\Sigma(X_j) \cup \Sigma((X_j)_\lambda)$ (for $j \in \{1,\ldots,6\}$), hence $D_{Z_i} = D_Z$.

Let $T = T_A \subset |2\Theta|_{00}$ be the variety swept by the l_X for $X \in P^{-1}(A)$. If $D : \mathcal{Z} \rightarrow |2\Theta|_{00}$ is the map which to a double solid Z associates D_Z, T is the image of D. By the above, D is constant on the orbits of the Cremona group (see 1.9). The following implies (among other things) that T is a threefold.

(5.2) **LEMMA** : *For* x *generic in* A *the hyperplane* $\tilde{h}(x) \in (|2\Theta|_{00})^*$ *(image by the* Γ_{00}*-map) contains exactly* 27 *lines* l_X .

Proof : The hyperplane $\tilde{h}(x)$ is the set of Γ_{00}-divisors that contain x . A line l_X is in $\tilde{h}(x)$ if and only if the Γ_{00}-divisors in l_X contain x , or if and only if

$$x \in [\Sigma(X) \cup \Sigma(X_\lambda)]$$

So by 4.6, for x generic, we have 27 such lines. Q.E.D.

(5.3) *Remarks* : 1) It can be seen directly, using the complete intersection technique of the proof of 3.5 i), that the double solids in the orbits of the Cremona group all have the same Γ_{00}-divisor of conics.

2) One can derive from 5.1 a proof of the fact that the tangent cone at 0 to $\Sigma(X)$ is (scheme theoretically) the Prym-canonical image of X .

As in 1.6, let $\mathcal{E} \subset A$ be the subvariety of points x such that $\Theta.\Theta_x = E_Z$ is the Fano of an element Z of $\mathcal{Z}_A$. Let $x \in \mathcal{E}$ be generic. As D_Z contains $\Sigma(X_i) \cup \Sigma((X_i)_\lambda)$, D_Z is the intersection point of the six lines l_i corresponding to X_i . The hyperplane $\tilde{h}(x)$ contains the lines l_i . Generically, the l_i's are not in a plane. Otherwise, the plane would contain also the l_{ij}'s . A hyperplane containing two of these planes contains too many lines. Hence $\tilde{h}(x)$ contains $\mathbb{P}T_xT$ and we have proved

(5.4) **PROPOSITION** : *At a point* x *such that* $\Theta.\Theta_x = E_Z$ *for a double solid* Z *, the hyperplane* $\tilde{h}(x)$ *is tangent to* T .

In the same way as 5.2, using 4.8, we obtain :

(5.5) **LEMMA** : *For* $x \in \mathcal{E}$ *generic, the hyperplane* $\tilde{h}(x)$ *contains exactly* 21 *lines* l_X , 6 *of which "count twice".*

We give a list of important corollaries of these below.

(5.6) **COROLLARY** : *Through a smooth point of* T *only* 6 *lines* l_X *pass.*

Proof : First notice that, generically, by 3.4, none of the l_{ij}'s passes through the intersection point of the l_i's . If there are more than six lines l_i passing through $D \in T$, then all these lines (and also the lines incident to two of them) have to be in the tangent space at D to T . As before the tangent hyperplane to T would contain too many lines.

(5.7) **COROLLARY** : *At a smooth point* D *of* T *, the curves* X_i *and* $(X_i)_\lambda$ *are the only Prym-curves whose* Σ*'s are in* D .

(5.8) **COROLLARY** : *Two generic curves* X *and* Y *in* $P^{-1}(A)$ *are tetragonally related if and only if* l_X *and* l_Y *intersect.*

Proof : If X and Y are tetragonally related, we can find a double solid Z such that $X = X_1$ and $Y = X_2$ then D_Z contains $\Sigma(X)$ and $\Sigma(Y)$ and is the intersection point of l_X and l_Y .

If l_X and l_Y intersect in D_Z , then by 5.7, after renaming, $X = X_1$ and $Y = X_2$ are discriminant curves for Z . By the proof of 2.6 and 1.9 they are tetragonally related.

(5.9) **COROLLARY** : *Two generic curves* X *and* Y *are tetragonally related if and only if* $\Sigma(X) \cap \Sigma(Y)$ *is of dimension* 1 *. For any fixed* X *the set of curves* Y *such that* X *and* Y *are tetragonally related or equivalently such that* $\Sigma(X) \cap \Sigma(Y)$ *is of dimension* 1 *is isomorphic to the inverse image* $\hat{Q}$ *of* $\tilde{Q}$ *in* $P^{-1}(A)$ *,* $\tilde{Q}$ *being the family of lines in* T *incident to* l_X *. (*$\hat{Q}$ *maps* 2-1 *to* $W^1_4(X)$ *= set of* g^1_4*'s on* X)

Proof : If X and Y are tetragonally related then there exists a double solid Z such that X and Y are the discriminant curves for the covers $B \longrightarrow \mathbb{P}^2$ when we project from two double points of Z . By 3.3, $\Sigma(X)$ and $\Sigma(Y)$ are contained in D_Z and their intersection is of dimension one .

Conversely suppose that $\Sigma(X) \cap \Sigma(Y)$ is of dimension 2. By 3.2 for any $a \in \Sigma(X) \cap \Sigma(Y)$ there are Prym-embeddings of $\tilde{X}$ and $\tilde{Y}$ in $\Theta.\Theta_a$. Hence, by 5.2, $\tilde{h}(a)$ contains l_X and l_Y and we have a one-dimensional family of hyperplanes in $\mathbb{P}^4$ (the projective space containing T) which contain l_X and l_Y : these two lines are in a plane so they intersect and by 5.8 X and Y are tetragonally related. Q.E.D.

(5.10) **COROLLARY** : *Suppose that the* Z_i *and* ιZ_i *(as in 3.5) are generic, then these are the only double solids with associated* Γ_{00}*-divisor* D_Z .

Proof: If Z is a double solid corresponding to D , then by 5.7, the discriminant curves for Z are X_i or $(X_i)_\lambda$ (these being the curves corresponding to the lines l_i through D).

Letting the Cremona group (see 2.1) act we can suppose that the curves X_i are the discriminant curves for the projections from the double points p_i of Z . It has been proved by Donagi and Clemens that the curves $X_2,\ldots, X_6$ determine the plane representation of X_1 . By 2.2 this implies that they determine the double solid Z . Our proof is close to Donagi's.

By 2.6, the curves $X_2,\ldots, X_6$ are tetragonally related to X_1 . Thus they determine 5 g^1_4's , say $g_2,\ldots,g_6$, on X_1 . These have the property

$$g_i + p_i' + p_i'' = g^2_6$$

where p_i', p_i'' are the points of X_1 above p_i and g^2_6 gives the plane representation of X_1 as discriminant curve for Z . Hence if $g^2_6 = K_X - p - q$, then

$$h^0(K_X - g_i - p - q) > 0$$

We see (cf. 1.3) that the singular quadrics q_i corresponding to the g_i's are on a line l in the net of quadrics containing the canonical model of X_1 . We need to show that the g_i's determine $\{p,q\}$ uniquely. This is a consequence of the following.

Let $\tilde{Q}' = W^1_4$ be the variety parametrizing the g^1_4's on $X = X_1$. Then (see for instance [ACGH]) JX is the Prym variety of the cover $\tilde{Q}' \to Q$. The g_i's give a lift of the divisor $D_l \in g^2_5$ cut on Q by l .

Recall the elementary fact about quadrics (of rank 4) that a line is contained in q_i if and only if it is contained in one of the rulings of the quadric. For a quadric q_i of rank 3 , a line is contained in q_i if and only if it is contained in its (unique) ruling.

Let S be the special subvariety of $P(\tilde{Q}',Q) = JX$ corresponding to g^2_5 , i.e., a translate of one component of the set

$$\{D \in \mathrm{Pic}^5\tilde{X} : h^0(D) > 0 \, , \, \pi_* D \in g_5^2 \}$$

(the two components of this are isomorphic, see [B3]). For each bisecant $\langle s,t \rangle$ to κX we can consider the five (counted with multiplicities) quadrics of rank ($\leq$) 4 containing κX and $\langle s,t \rangle$. Then $\langle s,t \rangle$ will be contained in one ruling of each quadric and hence will pick a g_4^1 above it. This defines a map $X^{(2)} \to S$ which is an isomorphism because S has the minimal homology class ([B3]) and the image of $X^{(2)}$ by this map is exactly its image by the Abel-Jacobi map. Q.E.D.

We know the Prym-embedded curves in a Fano variety E_Z associated to a double solid Z (4.8). For a generic $x \in A$, we want to describe the Prym-embedded curves in $\Theta.\Theta_x$.

(5.11) **LEMMA** : *Suppose that* X *and* Y *(smooth) are tetragonally related by* g_4^1 *on* X. *Then*

$$\Sigma(X) \cap \Sigma(Y) \;=\; \{[p,q] \, ; \, h^0(g_4^1 - \pi p - \pi q) > 0\}$$

Proof : Let $S = \{[p,q] \, ; \, h^0(g_4^1 - \pi p - \pi q) > 0\} \subset \Sigma(X)$ and denote by the same symbol the corresponding subvarieties of $X \times X$, $X^{(2)}$ and $\tilde{X}^{(2)}$.

As in 4.4, the homology class of S in $X \times X$ is $4E+4F - \Delta$. So the homology class of S in $\tilde{X}^{(2)}$ is $8\tilde{F} + 8\tilde{E} - \tilde{\Delta} - \tilde{\Delta}'$ ($\tilde{\Delta}$ is the diagonal, $\tilde{\Delta}'$ is the set of $\{p, \sigma p\}$ and $\tilde{E}$, $\tilde{F}$ are the two fibers in $\tilde{X}^{(2)}$). Hence in A, as the map from $S \subset \tilde{X}^{(2)}$ to A is of degree 2 onto its image (still denoted by S), the (reduced) homology class of S is

$$1/2 \, (16\Theta^3/3 - 4\Theta^3/3) \;=\; 2\Theta^3$$

(the embedding of a fiber in A gives us a Prym-embedding of $\tilde{X}$ with homology class $\Theta^3/3$, $\tilde{\Delta}$ has $2^2 = 4$ times the homology class of a Prym-embedded copy of $\tilde{X}$ because multiplication by 2 has degree 4 and $\tilde{\Delta}$ has dimension 1 , finally $\tilde{\Delta}'$ is blown down to 0).

From the fact that $\Sigma(X) \cup \Sigma(X_\lambda)$ is the base locus of the pencil of Γ_{00}-divisors

l_X we deduce that the homology class of $\Sigma(X) \cap \Sigma(Y)$ is also $2\Theta^3$. By 2.5, $S \subset \Sigma(X) \cap \Sigma(Y)$, hence we have equality.

(5.12) **LEMMA** : *If three generic lines* l_X, l_Y, l_U *are in a plane section of* T *, then either* $\{X,Y,U\}$ *or* $\{X_\lambda,Y_\lambda,U_\lambda\}$ *is a tetragonal triple.*

Proof : By 5.8, X is tetragonally related to Y and U. Let g_1 and g_2 be the g_4^1's relating X to Y and U respectively. The one-dimensional family of hyperplanes in $|2\Theta|_{00}$ is the image by $\tilde{h}$ of

$$(\Sigma(X) \cup \Sigma(X_\lambda)) \cap (\Sigma(Y) \cup \Sigma(Y_\lambda)) \cap (\Sigma(U) \cup \Sigma(U_\lambda))$$

The intersection of this with $\Sigma(X)$ is

$$\Sigma(X) \cap (\Sigma(Y) \cup \Sigma(Y_\lambda)) \cap (\Sigma(U) \cup \Sigma(U_\lambda))$$

Let $\quad S_i = \{[p,q] \,;\, h^0(g_i - \pi p - \pi q) > 0\} \subset \Sigma(X)$

and $\quad T_i = \{[p,q] \,;\, h^0(K_X - g_i - \pi p - \pi q) > 0\} \subset \Sigma(X)$

By 5.11 and 3.1, the above is equal to $(S_1 \cup T_1) \cap (S_2 \cup T_2)$. This being of pure dimension 1, we must have

either $S_1 = S_2$, $T_1 = T_2$

or $\quad S_1 = T_2$, $T_1 = S_2$.

In the first case, $g_1 = g_2$ and $\{X,Y,U\}$ is a tetragonally related triple. In the second case, $g_1 = K_X - g_2$ and $\{X,Y,U_\lambda\}$ is a tetragonally related triple ; changing the g_4^1 on U_λ to its opposit, one sees that this is equivalent to : $\{X_\lambda,Y_\lambda,U_\lambda\}$ is a tetragonally related triple. Q.E.D.

(5.13) Let a be a generic element of A. Let $\tilde{X}_\lambda$ be a Prym-embedded curve in $\Theta.\Theta_a$, then by 3.2 $a = [p,q]$ for some $p, q \in \tilde{X}$. If g_i $(i = 1,\ldots,5)$ are the 5 g_4^1's such that $h^0(g_4^1 - \pi p - \pi q) > 0$ (as in the proof of 5.10) and (Y_i,U_i) is the pair of curves tetragonally related to X via g_i, then by 2.5 and 3.2 we have embeddings of $(\tilde{Y}_i)_\lambda$, $(\tilde{U}_i)_\lambda$ in $\Theta.\Theta_a$: this gives us 11 curves in $\Theta.\Theta_a$ and 11 lines in $H = \tilde{h}(a)$.

We claim that for a generic, no three lines of the form l_X in H pass through the same point. Otherwise, as H is not tangent to T, the three lines will have to be in a plane. By 5.12, the three lines correspond to three tetragonally related curves Y, Y', Y''. Considering the hyperplane tangent to T at the intersection point of l_Y, $l_{Y'}$ and $l_{Y''}$, we see that *generically* this is not possible (use 3.4).

Now considering Y_1 one can repeat the same procedure and obtain eight *new* (by 5.12 and the fact that no three lines in H have a common intersection point) lines and Prym-embedded curves for $\Theta.\Theta_a$. Then we obtain again eight *new* lines and Prym-embedded curves for $\Theta.\Theta_a$ with U_1 instead of Y_2.

So we have *all* the Prym-embedded curves in $\Theta.\Theta_a$ together with their corresponding lines (11+8+8). Notice that, if we fix l_X, l_Y and l_U, all the other lines corresponding to Prym-curves in any hyperplane containing them encounter one of them. Also l_X, l_Y, l_U are the only lines (corresponding to Prym-curves) in their plane.

Consider Y_2 : as before we have eight lines in four planes containing l_{Y_2} and different from l_X, l_{Y_i}, l_{U_i}. However we have only 27 curves in $\Theta.\Theta_a$: these lines must occur already for Y_1 or U_1. By symmetry 4 of them must be in common with Y_1, the 4 others in common with U_1. So we know 5 lines in T intersecting simultaneously Y_1 and Y_2. The same is true for the other pairs of lines in H.

We are now ready to prove :

(5.14) **THEOREM** : T *is a cubic threefold.*

Proof : Consider a generic tetragonal triple $\{X,Y,U\}$. Then (5.12) the lines l_X, l_Y, l_U are in a plane V. We are going to show that $V \cap T$ is the union of these three lines, each occuring with multiplicity 1 .

Suppose that there is a point $t \in V \cap T$ which is not on $l_X \cup l_Y \cup l_U$. Let l be a line through t, corresponding to some Prym-curve. Let H be the hyperplane spanned by l and V. Then by 5.13 all lines (corresponding to Prym-curves) in H encounter one of the lines l_X, l_Y, l_U. Hence l is in V. However V contains only l_X, l_Y, l_U

(5.13).

Take a point $t \in l_X \setminus (l_Y \cup l_U)$. The tangent space to T at t contains l_X , five other lines through t and the lines incident to two of these (5.5). None of these 21 lines is in V except l_X (5.13). If $T_t(T)$ contains V , it contains too many lines. Q.E.D.

(5.15) **PROPOSITION** : *The only base point of* $|2\Theta|_{00}$ *is* 0 *if* A *is sufficiently generic.*

Proof : Let $S = \{[p,q] \; ; \; h^0(g^1_4 - \pi p - \pi q) > 0\}$ and $T = \{[p,q] \; ; \; h^0(K_X - g^1_4 - \pi p - \pi q) > 0\}$, then by 5.11 $S = \Sigma(X) \cap \Sigma(Y)$.

Suppose that $x \in A$ is a base point of $|2\Theta|_{00}$ distinct from the origin. Then, for all $X \in P^{-1}(A)$, we must have $x \in \Sigma(X)$ or $x \in \Sigma(X_\lambda)$. By the irreducibility of $P^{-1}(A)$ for A generic this implies

$$\text{for all } X \in P^{-1}(A), \quad x \in \Sigma(X)$$

and, in particular, choosing a generic curve X in $P^{-1}(A)$, $x \in \Sigma(X) \cap \Sigma(Y)$ for all curves Y tetragonally related to X . So for all $X \in P^{-1}(A)$:

for all g^1_4's on X : $h^0(g^1_4 - \pi p - \pi q) > 0$.

Thus the line $\langle \pi p, \pi q \rangle$ in the canonical space of X is contained in all singular quadrics containing the canonical model of X .

However ([ACGH]) the intersection of these quadrics is X unless X is trigonal in which case A is the jacobian of a curve. Q.E.D.

(5.16) **PROPOSITION** : T *is a cubic threefold with at most isolated double points as singularities.*

Proof : Suppose that the cubic threefold T has a triple point t . Then T is the cone of vertex t over an irreducible cubic surface S . The family of lines in T has the following components : lines through t and lines in the planes projecting to lines in S . The lines l_X corresponding to Prym-curves cannot be all in a plane because otherwise the pencil of hyperplanes containing the plane will be a line in $(|2\Theta|_{00})^*$ whose inverse image in A will be contained in $\Sigma(X) \cup \Sigma(X_\lambda)$ for every Prym-curve X . This is impossible because then $|2\Theta|_{00}$ would have a base point besides 0 (the lines l_X generate $|2\Theta|_{00}$ by 5.14). Hence the lines l_X all pass through t. Then the divisor corresponding to t contains $\Sigma(X) \cup \Sigma(X_\lambda)$ for all X. However the map $\Sigma_A = (\cup \Sigma(X)) \rightarrow A$ is surjective by 4.6.

Suppose now that T has a double curve C. Then every hyperplane section of T will be singular. Also, as T is irreducible, a general hyperplane section of T will contain a finite number of lines and as it is singular it will contain less than 27 lines. This is ruled out by 5.2. Q.E.D.

Consider a generic line $l_X \subset T$. The projection from l_X exhibits T as a conic bundle over $\mathbb{P}^2$. The discriminant curve in $\mathbb{P}^2$ for this conic bundle is the image of the set of plane sections of T that are unions of three lines. By 5.12, 3.1 and the uniqueness of the plane representation of a plane quintic, this is projectively isomorphic to the plane quintic Q parametrizing singular quadrics through the canonical model κX of X (similarly for X_λ). Generically on $P^{-1}(A)$ the discriminant curve Q is smooth, so:

(5.17) **PROPOSITION**: T *is smooth for a generic abelian variety.*

We will determine the degree, the ramification locus and the branch locus of $\tilde{h}$ and h. For this, we will give a precise description of the fibers of $\tilde{h}$. This description will help us to relate directly $|2\Theta|_{00}$ to the spaces of quadrics through the canonical model of X. The relation between a Γ_{00}-divisor D and the quadric q associated to it is geometrically expressed in terms of bisecants of κX lying in q.

(5.18) **PROPOSITION**: *The degree of* $\tilde{h}$ *is* 2^7.

Proof:

(5.19) Choose a smooth curve $X \in P^{-1}(A)$. Consider the net N of quadrics containing the canonical embedding of X with the plane quintic Q corresponding to singular quadrics. Let $\psi : X^{(2)} \longrightarrow N^* \cong (\mathbb{P}^2)^*$ be the morphism associating to $\{p,q\}$ the line l_{pq} in N consisting of all quadrics containing the line $\langle p,q \rangle$ in $|K_X|^*$. Take a line L in

N which is *not in the image of the diagonal* by ψ, is not tangent to Q and does not pass through any double point of Q. Let Y and U be tetragonally related to X through two g_4^1's g_1 and g_2 which project to two distinct points on $Q \cap L$. Then, by 5.17, l_Y, l_U span a hyperplane H in $|2\Theta|_{00}$ with $l_X \subset H$.

(5.20) Notice that

$$\tilde{h}^{-1}(H) = [\Sigma(Y) \cup \Sigma(Y_\lambda)] \cap [\Sigma(U) \cup \Sigma(U_\lambda)] \quad \subset \quad \Sigma(X) \cup \Sigma(X_\lambda)$$

(see the proof of 5.2). Suppose that $h([p,q]) = H$ with $[p,q] \in \Sigma(X)$, then (see 5.13) the five pairs of lines incident to l_X in H project in N to the intersection points of l_{pq} (= $l_{\pi p,\pi q}$) with Q. As the lines l_Y and l_U are among these lines we must have for $i = 1$ and 2 :

either $h^0(g_i - \pi p - \pi q) > 0$

or $h^0(K_X - g_i - \pi p - \pi q) > 0$.

That is, in the canonical space of X, the line spanned by πp, πq is in the intersection S of the singular quadrics q_i corresponding to g_i for $i = 1, 2$.

There are 16 lines in S that are all bisecant to κX. Hence we have $17.4 = 64 = 2^6$ distinct points in $\tilde{X}^{(2)}$ that project to these. By assumption 5.19, these project to 2^6 distinct points in $\Sigma(X)$ that are distinct from 0. Counting the points in $\Sigma(X) \cup \Sigma(X_\lambda)$ we obtain $3.2^6 = 2^7$ distinct preimages for H apart from the origin.

Now the proposition follows from proposition 5.15. Q.E.D.

(5.21) **DEFINITION** : We define *the multiplicity of* $|2\Theta|_{00}$ *at* 0 in the following way :

For a generic subsystem H of codimension 1 of $|2\Theta|_{00}$ the base locus of H is a finite set. This is the union of 0 and a set of distinct points, distinct from 0. The point 0 occurs with a certain multiplicity in the intersection of the elements of H : more precisely this is the length of the maximal Artinian subscheme of A with underlying set 0 and contained in every element of H. This length is an upper-semicontinuous function on $(|2\Theta|_{00})^*$ and is constant on a nonempty open subset of $(|2\Theta|_{00})^*$. The multiplicity at 0 for H generic is the multiplicity of $|2\Theta|_{00}$ at 0.

(5.22) *Remark* : It is clear from the above definition that the sum of the multiplicity at 0 of $|2\Theta|_{00}$ and the number of points distinct from 0 in the base locus of a generic H is equal to the number of points in the base locus of a generic subsystem of (projective) dimension 3 of $|2\Theta|$. Hence the previous results imply in particular that the multiplicity at 0 of $|2\Theta|_{00}$ is 4^4 because (see 5.2) the inverse image of a generic hyperplane H is scheme-theoretically :

$$[\Sigma(Y) \cup \Sigma(Y_\lambda)] \cap [\Sigma(Z) \cup \Sigma(Z_\lambda)]$$

the above intersection multiplicity is $(2\Theta)^4 = 2^8 \cdot 3 = 2^7 + 4^4$.

For each Γ_{00}-divisor D, let B(D) be the projectivised tangent cone to D at 0. Recall that $\tau|2\Theta|_{00}$ is the linear system generated by the *quartic* tangent cones B(D).

(5.23) **COROLLARY** : *If* A *is sufficiently generic, the base locus of* $\tau|2\Theta|_{00}$ *is empty. In particular,* $\tilde{h}$ *is a morphism.*

Proof : Consider four generic Γ_{00}-divisors $D_1, \dots, D_4$. Let $b : \tilde{A} \to A$ be the blow up map. Let $\tilde{D}_i$ be the strict transform of D_i in $\tilde{A}$.

We have to show that $\tilde{D}_1 \cap \dots \cap \tilde{D}_4 \cap E$ is empty and that the D_i's all have multiplicity 4 at 0. For this we will show that if this is not the case, then the multiplicity of $\tilde{D}_1 \cap \dots \cap \tilde{D}_4$ at 0 is greater than 4^4 : this contradicts 5.22.

The multiplicity at 0 of $D_1 \cap \dots \cap D_4$ is given by the multiplicity of 0 in the cycle

$$b_* \{(b^*D_1).(b^*D_2).(b^*D_3).(b^*D_4)\} .$$

The D_i's being generic, they all have the same multiplicity 2n at 0 with $n \geq 2$. Writing $b^*D_i = 2nE + \tilde{D}_i$ and developing we obtain

$$(b^*D_1).(b^*D_2).(b^*D_3).(b^*D_4) = \tilde{D}_2.\tilde{D}_3.\tilde{D}_4.\tilde{D}_4 + \Sigma_{1\leq i<j<k\leq 4} (2nE).\tilde{D}_i.\tilde{D}_j.\tilde{D}_k$$
$$+ \Sigma_{1\leq i<j\leq 4} (2nE)^3.\tilde{D}_i.\tilde{D}_j + \Sigma_{1\leq i\leq 4} (2nE)^4.\tilde{D}_i + (2nE)^4$$

If $\tilde{D}_1 \cap \dots \cap \tilde{D}_4 \cap E$ is nonempty, then $\tilde{D}_2.\tilde{D}_3.\tilde{D}_4.\tilde{D}_4$ will have a positive contribution to the multiplicity at 0, say a (because $\tilde{D}_i.E = \mathcal{O}_{\mathbb{P}^3}(4)$ is positive, see [F] p. 218). Summing up the multiplicities at 0 we get

$$a + 5.(2n)^4 - 7.(2n)^4 + 5.(2n)^4 - (2n)^4 = a + (2n)^4$$

By 5.22, this has to be equal to 4^4. Q.E.D.

(5.24) As the discriminant curve for the projection of T from l_X is Q (see 5.16), the space of hyperplanes in $|2\Theta|_{00}$ that contain l_X can be identified with the dual space of the net $N \supset Q$ parametrizing quadrics through κX. By the description of the fibers of $\tilde{h}$ in 5.20, the restriction of $\tilde{h}$ to $\tilde{X}^{(2)} \subset \tilde{A}$ factors through the map $\psi : X^{(2)} \to N^*$. For each pencil of quadrics $l \in N^*$, $\tilde{h}^{-1}(l)$ is the inverse image in $\tilde{X}^{(2)}$ of the bisecants to κX that are in the intersection of the quadrics of l. Hence for each line in N^*, corresponding to a quadric q in N, $\tilde{h}^{-1}(q)$ is the inverse image in $\tilde{X}^{(2)}$ of the set of bisecants to κX that are in q. A line in N^* is a pencil of lines in N. As we have the projection from l_X:

$$|2\Theta|_{00} \to N$$

by inverse image, a pencil of lines in N will give a pencil of hyperplanes in $|2\Theta|_{00}$ containing a fixed plane V in $|2\Theta|_{00}$ such that $V \supset l_X$. The intersection of a Γ_{00}-divisor which is a generic element of V with $\Sigma(X)$ is exactly $\tilde{h}^{-1}(q)$. This gives a geometric way of associating a quadric $q_D \in N$ to a Γ_{00}-divisor D. Hence a geometric interpretation of the projection from l_X onto N.

We are now ready to determine the branch locus of $\tilde{h}$, we will show :

(5.25) **PROPOSITION** : *The branch locus of* $\tilde{h}$ *(resp.* h *) has two components* : $R_0 = \tilde{h}(E)$ *and* T^*, $h^{-1}(R_0)$ *is the union of the diagonals of the* $\Sigma(X)$ $(X \in P^{-1}(A))$.

Proof : By 5.23, $\tilde{h}$ is a morphism. Thus the branch locus of $\tilde{h}$ is a divisor. From the description of the inverse image of a hyperplane H under h given in 5.20 we deduce that $\tilde{h}$ can branch in three ways :

i) one of the inverse images x of H lies in $\Sigma(X) \cap \Sigma(X_\lambda)$,

ii) two of the lines (bisecant to the canonical model of X) in the intersection of the quadrics in l_{pq} (= S = $q_1 \cap q_2$ in 5.20) coincide,

iii) one of the inverse images of H is in the diagonal $\Delta(X)$ of $\Sigma(X)$.

In case i) X and X_λ are Prym-embedded in $\Theta.\Theta_x$. As curves correspond to lines and X and X_λ correspond to the same line, the number of lines in $T_H = T \cap H$ is less than 27 and H is tangent to T.

In case ii) also H is tangent to T:

The quartic Del Pezzo surface S contains 16 distinct lines unless it is not smooth, in the generic singular case (one double point) four of its lines count twice and pass through the double point t (see [Dm]). The quadrics in l_{pq} all have the same Zariski tangent space at t so at least one of them is singular at t; as Q parametrizes the singular quadrics this has to be one of the intersections points of Q with l_{pq} say q_1. Taking $\langle \pi p, \pi q \rangle$ to be one of the lines through t:

$$h^o(g_1 - \pi p - \pi q) > 0 \quad \text{and} \quad h^o(K_X - g_1 - \pi p - \pi q) > 0$$

Then l_{pq} is tangent to Q at q_1 and if Y is tetragonally related to X via g_1, $\Theta.\Theta_x$ contains Prym-embeddings of $\tilde{Y}$ and $\tilde{Y}_\lambda$: as in i) H is tangent to T.

Case iii) corresponds to the second component of the branch locus.

Generically on this component we see that $\tilde{h}^{-1}(H)$ has $2^7 - 2$ elements in $\tilde{A} \setminus E$ and one element in E that counts with multiplicity two. Q.E.D.

(5.26) *Remark* : Notice that in case i) above l_X is one of the lines in $T_H = T \cap H$ passing through the singular point and that in case ii) l_X does not pass through the singular point of T_H.

Next we wish to prove the assertions of 1.6 about the inverse image of T* in A and to give a characterization of $\mathfrak{E}$.

Looking back at the proof of 5.25 and at 5.26, we see that, taking l_X to be one of the lines which do not contain the double point of T_H, $h^{-1}(H) \cap \Sigma(X)$ has (generically)

48 distinct elements, 16 of them correspond to the four lines in S which pass through the double point t of S and in fact the 32 others are not ramification points of h, but if $x \in R'$ is one of these points, for each line l_Y through the double point of T_H, $\Theta.\Theta_x$ contains only one of the Prym-curves corresponding to it : $\Theta.\Theta_x$ *contains* 21 *Prym-embedded curves, six of them counting twice.* So we conclude :

(5.27) **PROPOSITION** : *If an intersection* $\Theta.\Theta_x$ *contains a Prym-embedded curve* $\tilde{X}$ *and* $\tilde{X}_\lambda$ *then it is the Fano variety of lines of a double solid.*

We wish to say a few words about points of order 2 . We first observe

(5.28) **LEMMA** : *The map* $1 : P^{-1}(A) \to F$ *, which to each Prym-curve* X *associates its line* l_X *, is a double cover.*

Proof : As A is generic, every Γ_{00}-divisor is irreducible. So the base locus of the pencil l_X is always a surface. (It is equal to $\Sigma(X) \cup \Sigma(X_\lambda)$ for X smooth.) So 1 is a morphism. Also 1 does not contract any curve. As the fano surface of lines F of T is smooth we have that 1 is also a finite morphism. Hence 1 is a double cover. Q.E.D.

(5.29) Fix a curve $X \in P^{-1}(A)$, let $Q \subset N$ be as before. As $l_X = l_{X_\lambda}$ (5.1) we deduce that N can also be identified with the net of quadrics containing κX_λ , and Q is also the plane quintic parametrizing singular quadrics containing κX_λ . Therefore, on JQ we are given three points of order 2 :

- α with associated double cover $\tilde{Q} \to Q$ and $P(Q,\alpha) = JT$
- α' with associated double cover $\mathrm{Sing}\Theta' \to Q$ and $P(Q,\alpha') = JX$
- α'' with associated double cover $\mathrm{Sing}\Theta'' \to Q$ and $P(Q,\alpha'') = JX_\lambda$.

Let $\hat{Q}$ be the inverse image of $\tilde{Q}$ in $P^{-1}(A)$, then, by 3.1, $\hat{Q}$ is a fiber product in two (distinct) ways :

$$\begin{array}{ccc} \hat{Q} & \longrightarrow & \tilde{Q} \\ \downarrow & & \downarrow \\ \mathrm{Sing}\Theta'' & \rightarrow & Q \end{array} \qquad\qquad \begin{array}{ccc} \hat{Q} & \longrightarrow & \tilde{Q} \\ \downarrow & & \downarrow \\ \mathrm{Sing}\Theta' & \rightarrow & Q \end{array}$$

The involution of $\hat{Q}$ for the cover $\hat{Q} \to \tilde{Q}$ is just λ. Let $\sigma_{\alpha'}$ and $\sigma_{\alpha''}$ be the involutions of $\hat{Q}$ for the covers $\hat{Q} \to \tilde{Q}' = \mathrm{Sing}\Theta'$ and $\hat{Q} \to \tilde{Q}'' = \mathrm{Sing}\Theta''$. As generically $X \neq X_\lambda$, $\sigma_{\alpha'} \neq \sigma_{\alpha''}$.

Let q be an element of Q. Let g_1 and h_1 be the two opposite g_4^1's corresponding to the two elements of $\mathrm{Sing}\Theta' \subset JX$ above q (see 1.3). Let Y and U be tetragonally related to X via g_1, then (3.1) Y_λ and U_λ are tetragonally related to X via h_1, i.e., $\sigma_{\alpha'}(Y) = U$ and $\sigma_{\alpha'}(Y_\lambda) = U_\lambda$. Similarly $\sigma_{\alpha''}(Y) = U_\lambda$ and $\sigma_{\alpha''}(Y_\lambda) = U$.

Therefore, the three involutions commute and the product of two of them is equal to the third one.

The above involutions are the nonzero elements of a group of automorphisms of $\hat{Q}$ isomorphic to $(\mathbb{F}_2)^2$.

Hence the points $\alpha, \alpha', \alpha''$ *commute and are the nonzero elements of a vector-subspace* V^2 *of* $(JQ)_2$.

The quadratic form on $(JQ)_2$ is given by $h^0(g_5^2 \otimes .)$ modulo 2. The antisymmetric bilinear form on $(JQ)_2$ is given by ([M2])

$$(\beta,\beta') \equiv h^0(g_5^2) + h^0(g_5^2 \otimes \beta) + h^0(g_5^2 \otimes \beta') + h^0(g_5^2 \otimes \beta \otimes \beta') \mod 2$$

By a straightforward computation we deduce that V^2 is totally isotropic with respect to $(,)$.

Therefore in the exact sequences

$$0 \to \{\alpha\} \to \{\alpha\}^\perp \to (JT)_2 \to 0$$
$$0 \to \{\alpha'\} \to \{\alpha'\}^\perp \to (JX)_2 \to 0$$
$$0 \to \{\alpha''\} \to \{\alpha''\}^\perp \to (JX_\lambda)_2 \to 0$$

associated to our Prym constructions ([M3]) V^2 projects to the three points μ, η and η_λ. The Prym variety of the last two is A. The first one gives by restriction the double cover $1 : P^{-1}(A) \to F$ where F is the variety of lines in T (by [CG] the albanese variety of F is isomorphic to JT). A consequence of the above discussion is

The point μ *of order* 2 *in* JT *determined by* A *is even.*

Here " μ *even* " is equivalent to the fact that μ is the image of even points of order 2 on JQ .

(5.30) Next we mention some geometric "pictures" that summarize the relations given by λ and the tetragonal construction.

Consider a plane section of T , union of three lines l_X , l_Y , l_U . By 5.12 we can suppose that {X,Y,U} is a tetragonally related triple. The three plane quintics Q, Q_1, Q_2 associated to these lines with their double covers $\tilde{Q}$, $\tilde{Q}_1$, $\tilde{Q}_2$ form a tetragonal triple :

Let $t \in Q$ be the common image of $l_Y = l(Q_1)$ and $l_U = l(Q_2)$ in N . Each line incident to l_Y picks via incidence a lifting in $\tilde{Q}$ of a divisor of g^2_5 - t ; similarly for lines incident to l_U .

The curves W' and W" tetragonally related to (SingΘ',Q) through g^2_5 - t are trigonal because the common Prym variety of the tetragonal triple is JX : by [B1] the only smooth curves with Prym variety a jacobian of dimension 5 are trigonal curves and plane quintics (there is only one plane quintic with Prym JX). The same is true for the tetragonal triple obtained from (SingΘ",Q) , and also for those obtained with Q_1 and Q_2 .

Donagi observes that the curves in every tetragonally related triple are trigonally related to a curve W (here of genus 7). The curve W comes with the three points β, β_1, β_2 of order 2 with respective Prym varieties JQ , JQ_1 , JQ_2 and with a totally isotropic $\mathbb{F}_2$-vector space V^3 of dimension 3 which is the common inverse image of the totally isotropic vector spaces in $(JQ)_2$ (called V^2 in 5.29), $(JQ_1)_2$, $(JQ_2)_2$ by the exact sequences analogous to those in 5.29 .

So we have seven curves of genus 6 whose jacobians are Pryms of this trigonal curve for the elements of $\mathbb{P}V^3$. Three of these are the plane quintics and the other 4 are the trigonals W', W" and their analogues. The three points β, β_1, β_2 are on a line in $\mathbb{P}V^3$. The same is true for every set of three points associated to a tetragonal triple. Hence each line is associated to an abelian variety of dimension 5 which is the common Prym variety of the curves on that line ; six of these are jacobians and the last one is JT ; they all come with a point of order 2 image of V^3 (these are μ and η's with our previous notations). The above was done by Donagi in reverse order (start with a trigonal W and V^3) : this is how he introduced (T,μ) for a generic ppav.

The space $\mathbb{P}V^3$ is a triangle.

The common Prym variety for the abelian varieties of dimension 5 in the configuration (in the more general sense of Mumford in [M3] "Prym varieties" can be defined for every abelian variety with certain data) is A .

The space V^3 is a cube. One of the vertices of the cube is the origin.

Now dualize the cube : we obtain another cube with vertices : the origin, T , X , Y , U , X_λ ,Y_λ , U_λ .

The projectivization of the cube is a triangle : $\mathbb{P}(V^3)^*$.

Embed $\mathbb{P}(V^3)^*$ in $\mathbb{P}^5(\mathbb{F}_2)$ by the Veronese map. Then project from T : we obtain an octahedron with vertices the six straight lines in the triangle or the six planes in the cube that are also planes in the euclidian sense. Each vertex corresponds to one of the curves X , Y , U , X_λ ,Y_λ , U_λ . Painting in white a face of the octahedron whose vertices are tetragonally related curves and painting the other faces alternatively in black and white, all white triangles correspond to tetragonally related triples and black triangles correspond to *non*tetragonally related triples.

See Figures 1 - 5 for the Triangle, the Cube, the dual of the Cube, the dual of the Triangle and the Octahedron respectively.

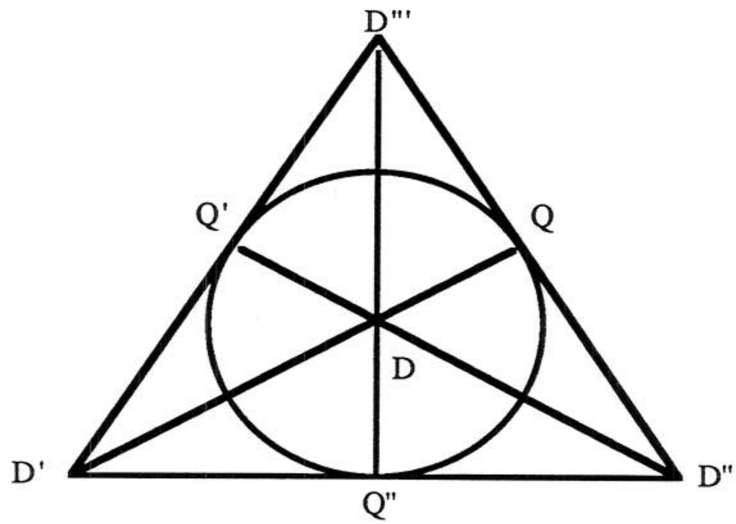


Figure 1 : The Triangle

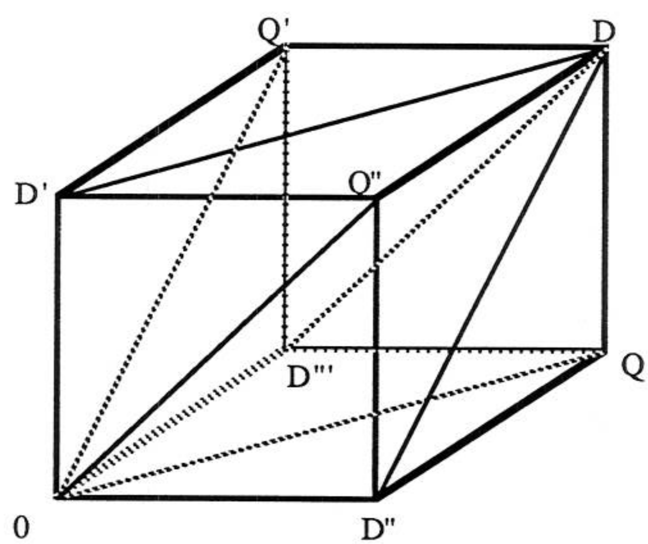


Figure 2 : The Cube

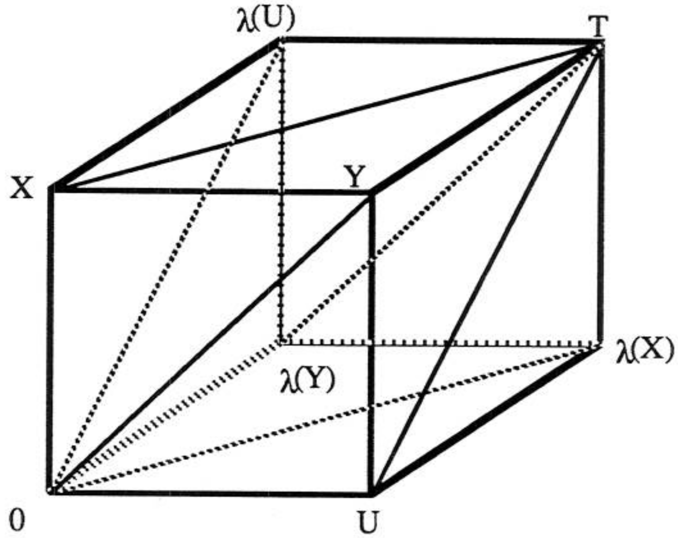


Figure 3 : The Dual of the Cube

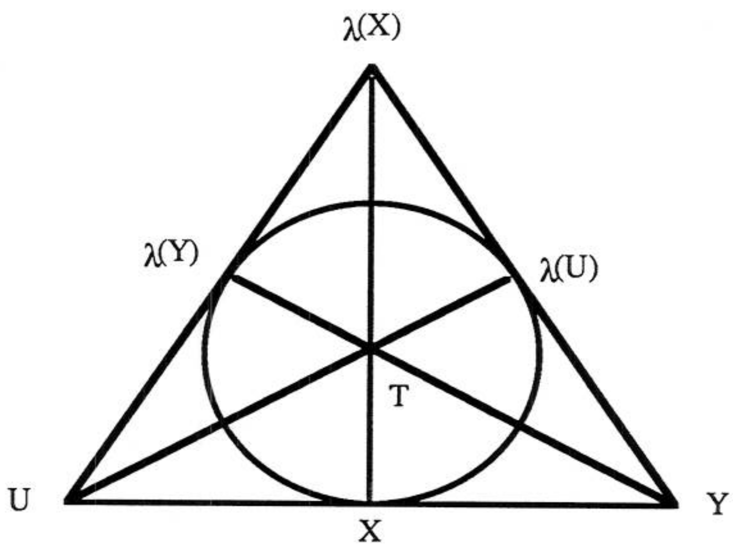


Figure 4 : The Dual of the Triangle

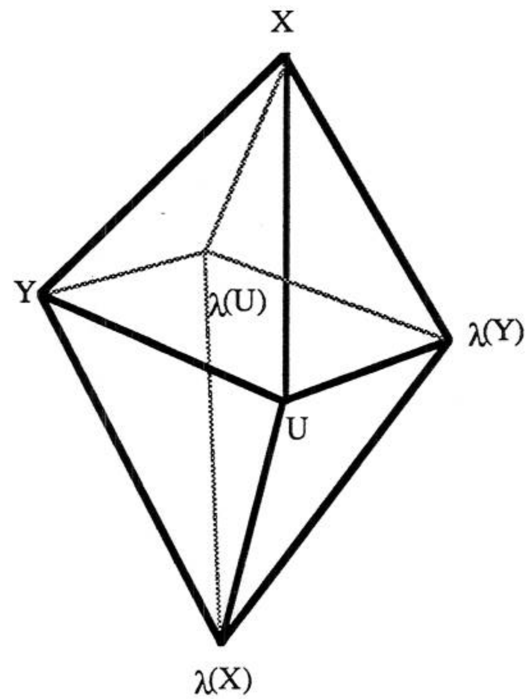


Figure 5 : The Octahedron

6. RELATION WITH PPAV'S OF DIMENSION 5

We first need some preliminaries on Prym-embeddings of curves in the theta divisor - denoted by $\overline{\Theta}$ - of a five-dimensional abelian variety B.

Let Y and U be two tetragonally related curves with B as common Prym variety. Let $g^1_4 \in W^1_4(U)$ relate U to Y and $g^2_6 = K_U - g^1_4$. Suppose that B is not a hyperelliptic jacobian and that, if U is bielliptic, g^2_6 is not the pullback of a g^2_3 on an elliptic curve.

(6.1) **LEMMA** : *The family* $\tilde{Y}''$ *of Prym-embeddings of* $\tilde{Y}$ *in* $\tilde{\Theta}$ *is a special subvariety associated to* g^2_6 ([B3]) *, i.e., one of the two components of*

$$\{E \in \mathrm{Pic}^6\tilde{U} : h^0(E) > 0 \, , \pi_* E \equiv g^2_6\}$$

so by [BD1]*its homology class is* $3.[\Theta]^3/_{3!} = [\Theta]^3/_3$.

Proof : Prym-embeddings of $\tilde{Y}$ are translates of one component of

$$\{E \in \tilde{U}^{(4)} : \pi_* E \equiv g^1_4\}$$

so they are given by divisors $G \in \mathrm{Pic}^6\tilde{U}$ such that $E + G \in \Theta$ for all $E \in \tilde{Y}$. This is equivalent to $\pi_{U*}G \equiv K_Z - \pi_{U*}E \equiv g^2_6$ and $h^0(E+G)$ is even and positive. This will be true for G in one of the two components of $\{E \in \mathrm{Pic}^6\tilde{U} : h^0(E) > 0 \, , \pi_* E \equiv g^2_6\}$.

The rest of the proof is totally similar to the proof of 2.4 . Q.E.D.

Let $A \in \mathcal{Q}_4$. By [M1] (page 227) the space of extensions of A by G_m is isomorphic to A via the composition

$$A \longrightarrow \mathrm{Pic}^0 A \longrightarrow \mathrm{Ext}^1(A, G_m)$$

where the first map is $x \longmapsto \mathcal{O}_A(\Theta - \Theta_x)$ and the second one is $L \longmapsto L \setminus o(A)$ where o is the 0-section of the bundle L.

For each $x \in A$, letting A_x be the extension of A with extension data x, we have a rational section $s : A \longrightarrow A_x$ obtained from the rational section of $L(\Theta - \Theta_x)$ which is 0 on Θ and ∞ on Θ_x. Composing s with $-\mathrm{id} : A \longrightarrow A$ we deduce that we have an isomorphism of extensions :

$$\begin{array}{ccccccccc} 0 & \to & G_m & \to & A_x & \to & A & \to & 0 \\ & & \mathrm{id}\downarrow & & \downarrow\cong & & \downarrow\text{-id} & & \\ 0 & \to & G_m & \to & A_{-x} & \to & A & \to & 0 \end{array}$$

This permits us to construct the $\mathbb{P}^1$-bundle $\tilde{A}_x$ over A by gluing the bundles $L(\Theta - \Theta_x)$ and $L(\Theta - \Theta_{-x})$ along $A_x \cong A_{-x}$. We obtain $\overline{A}_x$ from $\tilde{A}_x$ by gluing the 0-section $A_0 \cong A$ and the ∞-section $A_\infty \cong A$ (in terms of a parameter which is 0 on the zero section of $L(\Theta - \Theta_x)$ for instance) identifying $a \in A \cong A_0$ with $a+x \in A \cong A_\infty$ ([M4]).

By [BC] the image of s in $\overline{A}_x$ is the limit of the theta divisors of a generic family of five-dimensional ppav's with central fiber $\overline{A}_x$. We will denote $\overline{\Theta}^x = s(A)$ (resp. $\tilde{\Theta}^x$, resp. Θ^x) in $\overline{A}_x$ (resp. $\tilde{A}_x$, resp. A_x).

Let $X \in P^{-1}(A)$ be smooth such that $x = [p,q] \in \Sigma(X)$. Let $X_{pq} = X_{/p=q}$ and $\tilde{X}_{pq} = \tilde{X}_{/p=\sigma q, \sigma p=q}$. We have

(6.2) **LEMMA** : (i) *There is a two-dimensional family of embeddings of* $\tilde{X}_{pq}$ *in* $\overline{\Theta}^x$ *parametrized by the set*

$$\{E \in \mathrm{Pic}^8\tilde{X} \,;\, \pi_*E \equiv K_X \,,\, h^0(E) = 1 \,,\, h^0(E - x) = 1 \,,\, h^0(E - \sigma p - \sigma q) > 0\}$$

These embeddings project to Prym-embeddings of $\tilde{X}$ *in* A.

(ii) *The one-dimensional families of Prym-embeddings of* $\tilde{X}$ *in* Θ *and their inverse images by* t_x *in* Θ_x *(which by intersection give us the 2 Prym-embeddings of* $\tilde{X}_\lambda$ *in* $\Theta.\Theta_x$ *) have the same image by* s *in* $\overline{A}_x$ *and give us a double curve in the boundary of the family in* (i) .

Proof : Parametrize A with X as in 2.4, then all Prym-embeddings of $\tilde{X}$ in A are of the form

$$j_E : \widetilde{X} \longrightarrow A$$
$$p \longmapsto p - \sigma p + E$$

where $E \in \mathrm{Pic}^8\widetilde{X}$ is such that $\pi_* E \equiv K_X$ and $h^0(E)$ is odd. As before $j_E(\widetilde{X}) = \widetilde{X}_E \subset \Theta$ (resp. Θ_x) if and only if $h^0(E) = 3$ (resp. $h^0(E+x) = 3$). ($h^0(E) \geq 5$ is excluded by Clifford's lemma)

So the first conditions on E are $h^0(E) = 1$ and $h^0(E - x) = 1$ because s sends Θ and Θ_x into A_0 and A_∞ in a 1-to-1 way.

Now $\quad j_E{}^*\Theta = \{p\,;\, h^0(p - \sigma p + E) > 0\}$

$\quad j_E{}^*\Theta_x = \{p\,;\, h^0(p - \sigma p + E - x) > 0\}$

So we get $\quad j_E{}^*\Theta = \{p\,;\, h^0(p + E) > 1 \text{ or } h^0(E - \sigma p) > 0\}$ and as $\sigma E + E \equiv K_X$

$\quad j_E{}^*\Theta = \{p\,;\, h^0(E - \sigma p) > 0\} = \sigma E$

similarly $\quad j_E{}^*\Theta_x = \{p\,;\, h^0(E - x - \sigma p) > 0\} = \sigma(E - x)$ (notice that as expected $j_E{}^*(\Theta_x - \Theta) = x$)

So we obtain in $\overline{A}_x$ a copy of $\widetilde{X}$ with 8 points identified unless some of the points are in $\Theta.\Theta_x$ (because $\Theta.\Theta_x$ is blown up by s), i.e., equal. We need

$$\sigma E = \Sigma_{1\leq i\leq 6}\, p_i + p_7 + p_8$$
$$\sigma E - \sigma x = \Sigma_{1\leq i\leq 6}\, p_i + q_7 + q_8$$

so $q_7 + q_8 - p_7 - p_8 = x$ that is $p_7 + p_8 = \sigma(p + q)$, $q_7 + q_8 = p + q$.

The rest of the assertions are now clear. Q.E.D.

(6.3) **COROLLARY** : A_x *is the Prym variety of the cover* $\widetilde{X}_{pq} \longrightarrow X_{pq}$.

Proof : By lemma 6.2 we have a surjective albanese map $\mathrm{alb} : J\widetilde{X}_{pq} \longrightarrow A_x$ with a commutative diagram :

$$\begin{array}{ccccc} \mathrm{alb} : J\widetilde{X}_{pq} & \rightarrow & A_x & \rightarrow & A \\ \downarrow & & & \nearrow & \\ J\widetilde{X} & \rightarrow & JX & & \end{array}$$

So the composition of the top horizontal arrows vanishes on pullbacks of divisor classes on X_{pq} which means that $\mathrm{alb}(\pi^*JX_{pq}) \subset \mathbb{C}^* = \mathrm{Ker}\,(A_x \longrightarrow A)$. So we have a group homomorphism

$$\mathbb{C}^* \longrightarrow JX_{pq} \longrightarrow \mathbb{C}^*$$

If it is nonzero, it has to be an isomorphism because it is induced by a linear homomorphism of $\mathbb{C}$ via the exponential map :

$$\begin{array}{ccc} T_1\mathbb{C}^* = \mathbb{C} & \rightarrow & \mathbb{C} = T_1\mathbb{C}^* \\ \exp.\downarrow & & \downarrow\exp. \\ \mathbb{C}^* & \rightarrow & \mathbb{C}^* \end{array}$$

Then the sequence

$$0 \longrightarrow \mathbb{C}^* \longrightarrow JX_{pq} \longrightarrow JX \longrightarrow 0$$

would split (x is generic). So our albanese map factors

$$\mathrm{alb}' : J\widetilde{X}_{pq} / JX_{pq} \twoheadrightarrow A_x$$

The Kernel of alb' is contained in

$$\mathbb{C}^* = (\mathbb{C}^*)^2/\mathbb{C}^* = \mathrm{Ker}\,(J\widetilde{X}_{pq} \longrightarrow J\widetilde{X}) \,/\, \mathrm{Ker}\,(JX_{pq} \longrightarrow JX)$$

and it is finite. However under alb' , $\mathbb{C}^*$ goes to $\mathbb{C}^*$. As before the restriction of alb' to $\mathbb{C}^*$ has to be an isomorphism. Q.E.D.

Hence the family in 6.2 is the family of Prym-embeddings of $\widetilde{X}_{pq}$ in $\overline{\Theta}^x$. We have

(6.4) **COROLLARY** : *Generically on* $\mathcal{E}xt\mathcal{A}_4$ *the family of Prym-embeddings of* $\widetilde{X}_{pq}$ *in* $\overline{\Theta}^x$ *is reduced.*

Proof : We need to consider the compactifications of the families in 6.2 that are the limits of the generic cycles. By 6.2 the underlying variety of our cycle for generic A_x is the image under s of

$$\{E \in \mathrm{Pic}^8\widetilde{X} \,;\, \pi_*E \equiv K_X \,,\, h^0(E) = 1 \,,\, h^0(E+x) = 1 \,,\, h^0(E - \sigma p - \sigma q) > 0\}$$

and the compactification is the image under s (in $\overline{A}_x$) of

$$\{E \in \mathrm{Pic}^8\tilde{X} \,;\, \pi_*E \equiv K_X \,,\, h^0(E) \textit{ odd} \,,\, h^0(E - \sigma p - \sigma q) > 0\}$$

By [B3] this has the same homology class as

$$\{E \in \mathrm{Pic}^8\tilde{X} \,;\, \pi_*E \equiv K_X \,,\, h^0(E) \textit{ even} \,,\, h^0(E - \sigma p - \sigma q) > 0\}$$

and by [BD1] this is equal to $\Theta.\Theta_{[p,q]}$, so has homology class $[\Theta]^2$ in A. Now with the notations of [BC] :

$$[\Theta]^2 = \Sigma_{1\le i<j\le 4}\ \gamma_i\times\delta_i\times\gamma_j\times\delta_j$$

and the homology class of the image in $\overline{A}_x$ is $3.[\Theta]^3/_{3!} = [\Theta]^3/_3$. Q.E.D.

Together with the above, the following gives a second proof of the result of [DS] , *namely, that the degree of* P *from* $\mathcal{P}_6$ *to* $\mathcal{A}_5$ *is* 27 .

(6.5) **PROPOSITION** : *The map* P *is generically unramified on* $\mathcal{G}en\mathcal{P}_6$.

Proof : The space $H^0(X_{pq},\omega_{pq}\otimes\Omega_{pq})$ is the cotangent space to the deformations of X_{pq} ([L] and [Sc]).

The codifferential of P is multiplication :

$$S^2H^0(X_{pq},\omega_{pq}\otimes\eta_{pq}) \longrightarrow H^0(X_{pq},\omega_{pq}\otimes\Omega_{pq})$$

where ω_{pq} is the dualizing sheaf of X_{pq} , Ω_{pq} is the sheaf of regular differentials on X_{pq} and η_{pq} is the point of order 2 associated to the cover $\tilde{X}_{pq} \to X_{pq}$.

If the Prym map is ramified at $(\tilde{X}_{pq},X_{pq})$, then there is a quadric q in $\mathbb{P}^4 = \mathbb{P}T_0A_x$ containing the Prym-canonical image χX_{pq} of X_{pq} . Let o be $\mathbb{P}T_0\mathbb{C}^* \subset \mathbb{P}^4$. Then o is the singular point of χX_{pq} and $o \in q$.

(6.6) *Claim* : *The quadric* q *is singular at* o .

Suppose that this is not the case and let $\mathbb{T} \cong \mathbb{P}^3$ be $\mathbb{P}T_0q$. Let

$$\nu \ :\ \mathbb{P}^4 = \mathbb{P}T_0A_x \ \longrightarrow\ \mathbb{P}T_0A = \mathbb{P}^3$$

be the projection from o . Consider the cone $\mathcal{C}X$ over the Prym-canonical image χX of X : this is a surface of degree 8 in $\mathbb{P}^4$, its intersection with q is a curve of degree 16 which contains χX_{pq} . Say $\mathcal{C}X \cap q = \chi X_{pq} + S$.

As o is the vertex of $\mathcal{C}X$, $\mathcal{C}X$ has a point of multiplicity 8 at o and S has a point of multiplicity 6 at o . However S is of degree 6 so its projection in $\mathbb{P}^3$ is a finite union of points : $S = \Sigma_{1\leq i\leq 6} l_i$ where l_i is a line through o . Let $p_i = \nu(l_i)$, then as $\mathbb{T}$ is a projective space of dimension 3 we have :

$$\Sigma_{1\leq i\leq 6}\, p_i + \pi p + \pi q \equiv K_X + \mu_X$$

so by the genericity assumption on p and q we can suppose that we have at least 5 distinct p_i's (or l_i's).

Choose a generic quadric in $\mathbb{P}^3$ containing the p_i's and let q' be the cone over it in $\mathbb{P}^4$. The intersection $q \cap q'$ is a quartic surface containing the l_i's and also $\mathbb{T} \cap q \cap q'$ contains the l_i's so, as we have at least five distinct l_i's

$$q \cap q' \subset \mathbb{T} \quad , \quad \nu(q') = \nu(q \cap q') = \nu(\mathbb{T}) \quad \text{is a plane.}$$

However $\nu(q')$ is a generic quadric containing the p_i's and it has rank greater than 2 .

Now, q being singular at o , $\nu(q)$ is a quadric in $\mathbb{P}^3$ containing κX , this means that $P : \mathcal{R}_5 \longrightarrow \mathcal{A}_4$ is ramified at X , but A is generic and this can not be.

Q.E.D.

Next we would like to determine the ramification locus of P (there is an unpublished proof of this in [Do3]).

By the analysis of h in section 5 (see especially the discussion preceding 5.27) we see that the number of curves in $\Theta.\Theta_x$ (hence the cardinality of the fiber of P at A_x) drops exactly when $x \in R'$. So we are looking for extensions A_x which contain six Prym curves whose corresponding lines intersect in one point on T . The first place to look for them is among generalized intermediate jacobians of double solids.

Let x be generic in R' and z be the point on T corresponding to h(x) . Let $(X_1)_\lambda, \dots, (X_6)_\lambda$ be the curves in $\Theta.\Theta_x$ which count with multiplicity 2 . Then if $x = [s_i,t_i] \in \Sigma(X_i)$, we have Prym-embeddings of $(\widetilde{X}_i)_{s_i t_i}$ in A_x .

Let Z be a double solid above z with double points $p_1,\dots, p_6$ such that the set of lines in Z through p_1 is $(\tilde{X}_1)_\lambda$ and the sets of lines in Z through $p_2, \ldots, p_6$ are respectively $\tilde{X}_2, \ldots, \tilde{X}_6$. Let $\overline{Z}$ be the blow up of Z at $p_2, \ldots, p_6$.

(6.7) **LEMMA** : *The extension data for the generalized intermediate jacobian* $J\overline{Z}$ *of* $\overline{Z}$ *is* $\pm x$.

Proof : First notice that there is no Prym-embedding of $(\tilde{X}_1)_\lambda$ (with two pairs of points identified) in $J\overline{Z}$, because such an embedding would have to be the image of $(\tilde{X}_1)_\lambda \subset E_{\overline{Z}}$ by AJ , but this curve has two singular points of multiplicity 6 (because we do not blow up p_1 in $\overline{Z}$). Letting y be the extension data for $J\overline{Z}$ we deduce that $y \in R'$.

Actually $y = \pm x$:

$\Theta.\Theta_x$ and $\Theta.\Theta_y$ contain the same Prym-curves and

by letting the Cremona group $\{i_{\mathfrak{F}}, \mathfrak{F} \subset \{p_1, \ldots, p_6\}\}$ act (see 1.2 and the proof of 3.5) we obtain 32 different sets of 6 curves with associated lines $l_{X_1}, \ldots, l_{X_6}$ so 64 different values for y (counting the negatives of the extension data's). Q.E.D.

By [B1] the generalized Prym mapping (still denoted by P) is proper on $\mathcal{A}_5$. Also $\mathrm{Pic}\mathcal{A}_5 \cong \mathbb{Z}$ by [SV] . Hence the divisor classes of all the components of the branch locus of P are ample. We infer that their closures in $\mathcal{A}_5 \cup \mathcal{E}xt\mathcal{A}_4$ must meet $\mathcal{E}xt\mathcal{A}_4$. Also the branching of P on $\mathcal{E}xt\mathcal{A}_4$ is simple along the locus of generalized jacobians of double solids because each ramification point counts twice and not more (generically). Thus the branch locus is irreducible and it can only be the closure of the locus of intermediate jacobians of double solids with five ordinary double points.

7. SCHOTTKY

In this section our aim is to prove what was announced in 1.7, namely, the main result will be that the base locus $V(\Gamma_{00})$ of $|2\Theta|_{00}$ is reduced to 0 if A is outside the locus $\overline{\mathfrak{J}_4} \cup \mathcal{A}_{n11}$. Notice that, for A generic, we proved this in proposition 5.15 . We define $\mathcal{A}_{n11}$ to be the locus of Prym varieties of $(\tilde{X},X)$ where

- either X is the union of two elliptic curves meeting in four points, (these ppav's are isogenous but not isomorphic to a product of two ppav's of dimension 2)

- or X is irreducible with one double point and its normalization has a vanishing theta-null. Furthermore, in this case we suppose that A has a two-dimensional family of such Prym-curves (in the first case this is automatically verified).

The locus $\mathcal{A}_{n11}$ has dimension 6.

Recall that $\overline{\mathfrak{J}_4}$ is the closure, in $\mathcal{A}_4$, of the locus of jacobians. In the following we will be using results of [B1] without mentioning it each time.

(7.1) **THEOREM** : *If* $A \notin \overline{\mathfrak{J}_4} \cup \mathcal{A}_{n11}$ *, then the generic curve in* $P^{-1}(A)$ *is smooth.*

Proof : As products of ppav's of lower dimension are in the closure of $\mathfrak{J}_4$, A is the (generalized) Prym variety of *connected* double covers $\tilde{X}$ of curves X of arithmetic genus 5 such that

(i) X is smooth or has at most nodes as singularities

(ii) the double cover $\tilde{X}$ is ramified exactly at the singular points of X

(iii) at the singular points of $\tilde{X}$ the two branches are not exchanged under the covering involution σ .

Then irreducible components of $\tilde{X}$ correspond to irreducible components of X .

Let $\tilde{X}_n$ and X_n be the normalizations of $\tilde{X}$ and X respectively. Let Ω_X be the sheaf of Kähler differentials on X. We have the exact sequence

(B) $$0 \to \oplus_i \mathbb{C} \to \Omega_X \to \omega_X \to \oplus_i \mathbb{C} \to 0$$

where ω_X is the dualizing sheaf of X ; the skyscraper sheaves have support on the singular locus of X. For each $p_i', p_i'' \in X_n$ projecting to a double point p_i of X, the sheaf on the right is generated by $(1/t_i', -1/t_i'')$ for some local coordinates t_i' and t_i'' at p_i' and p_i''. The sheaf on the left is generated by $t_i'dt_i'' = -t_i''dt_i'$. The tangent space at X to the space of deformations of X is $\mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$ (global Ext) : this has dimension 12. We have an exact sequence

(C) $0 \to H^1(X, \mathrm{Hom}(\Omega_X, \mathcal{O}_X)) \to \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X) \to H^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X) \to 0$

where $\mathcal{E}xt^1$ is the local Ext : a skyscraper sheaf supported on the singular locus of X (see [L] and [Sc]). The space $H^1(X, \mathrm{Hom}(\Omega_X, \mathcal{O}_X))$ is the tangent space at X to the space of deformations of X that are at least as singular as X, if s is the number of singular points of X, the dimension of this space is $12 - s$. Using Serre Duality, we see that the cotangent space to the space of deformations of X is $H^0(X, \Omega_X \otimes \omega_X)$.

Tensor the sequence (B) with ω_X, we obtain a cohomology sequence :

$$0 \to \oplus_i \mathbb{C} \to H^0(X, \Omega_X \otimes \omega_X) \to H^0(X, \omega_X \otimes \omega_X)$$

Let $\mathbb{T}^*$ be the image of $H^0(X, \Omega_X \otimes \omega_X)$ in $H^0(X, \omega_X \otimes \omega_X)$. Then the exact sequence

$$0 \to \oplus_i \mathbb{C} \to H^0(X, \Omega_X \otimes \omega_X) \to \mathbb{T}^* \to 0$$

is the dual of the sequence (C) and $\mathbb{T} = H^1(X, \mathrm{Hom}(\Omega_X, \mathcal{O}_X))$. The skyscraper sheaf is the vector space of sections of $\Omega_X \otimes \omega_X$ with support on the singular locus of X.

In the canonical space of $\tilde{X}$ the double points of $\tilde{X}$ are in the (-1)-eigenspace of the involution of $|K_{\tilde{X}}|^*$ induced by σ (because they are fixed by σ and project to the double points of X, hence cannot be in the (+1)-eigenspace). Thus the Prym-canonical image χX of X is the union of χX_n (curve of degree $8 - s$ and genus $5 - s$) and the lines connecting p_i' to p_i''.

From this and also [B1] (p. 170) we see that we can replace X by the union of X_n and $\mathbb{P}^1$'s joining p_i' to p_i''. This gives a flat family of curves over $\mathcal{P}_5$ and we have a morphism from this family to the bundle over $\mathcal{A}_4^*$ with fiber $\mathbb{P}T_0A$ at A.

For X smooth, the codifferential of P is multiplication

$$S^2H^0(X,\omega_X\otimes\eta) \rightarrow H^0(X,(\omega_X)^2)$$

as the codifferential of P is a bundle map from the cotangent bundle of $\mathcal{A}_4^*$ to the cotangent bundle of $\mathcal{P}_5$, we deduce that it is multiplication everywhere. It also follows from the above considerations that the kernel of the codifferential of the restriction of P to the space of curves that are at least as singular as X is the space of quadrics in T_0A which contain χX_n.

Consider the ramified cover $\pi_n : \tilde{X}_n \rightarrow X_n$. The variety $P(\tilde{X}_n,X_n)$ is isogenous to $P(\tilde{X},X)$. Let δ be the divisor class on X_n such that

$$\pi_{n*}\mathcal{O}_{\tilde{X}_n} \cong \mathcal{O}_{X_n}\oplus\mathcal{O}_{X_n}(\delta)$$

and let Δ be the ramification divisor of π_n ($\Delta \equiv 2\delta$), then the space of quadrics containing χX_n is also the kernel of multiplication

$$S^2H^0(X,\omega_X\otimes\eta) = S^2H^0(X_n,\omega_{X_n}(\delta)) \rightarrow H^0(X_n,(\omega_{X_n})^2(\Delta))$$

(Notice that we also obtain an isomorphism

$$H^0(X_n,(\omega_{X_n})^2(\Delta))^* \cong H^1(X,Hom(\Omega_X,\mathcal{O}_X)) \;)$$

(7.2) First suppose that A is simple, i.e., does not contain any abelian subvariety of smaller dimension, or equivalently, is not isogenous to a product of smaller dimensional abelian varieties ([M1] p. 173). Then every curve X in $P^{-1}(A)$ is irreducible. Also, as A is not in $\overline{\mathcal{J}}_4$, $\tilde{X}$ is *irreducible*.

For X singular we have the following possibilities :

genus of X_n	# of double points of X	genus of $\tilde{X}_n$	dimension of moduli of $(\tilde{X},X)$	the generic Q paramet - rizing singular quadrics containing κX	image by P
4	1	8	9+2 = 11	1 double point (irreducible)	$\mathcal{a}_4$
3	2	7	6+4 = 10	2 d. p. (irred.)	$\mathcal{a}_4$
2	3	6	3+6 = 9	3 d. p. (irred.)	
1	4	5	1+8-1 = 8	4 d. p. (irred.)	
0	5	4	10-3 = 7	5 d. p. (irred.)	ppav's isogenous to a hyperelliptic jacobian by an isogeny of degree 16

We are going to show that in each case the family of singular curves of the given type in $P^{-1}(A)$ is at most 1-dimensional.

Consider the restriction P_s of P to the space $\mathcal{P}_{6,s}$ of irreducible singular curves with exactly s double points.

Let us first consider the case where X has one double point. The moduli space of such curves is of codimension 1 in $\mathcal{P}_6$. We have to show that χX_n is not contained in any quadric.

If χX_n is contained in a nonsingular quadric q , write $X_n \equiv aL_1+bL_2$ in q , L_1 and L_2 being the two rulings of the quadric. We must then have (a-1).(b-1) = arithmetic genus of χX_n and a+b = 7 = degree of χX_n. Hence up to a transposition of a and b we have two cases :

- a = 2 and b = 5 . In this case X_n is hyperelliptic. The ppav A is the Prym of a double cover of X_n ramified at two points : by [B1] (p. 171), A is a jacobian.

- a = 3 and b = 4 . This is a possible case. In this case the arithmetic genus of χX_n is 6 and χX_n has two double points. Let ξ and g_0 be respectively the g^1_3 and the g^1_4 cut on X_n by the rulings of q. Then

$$K_{X_n}+\delta \equiv g_0+\xi$$

Let $\{u',u''\}$ and $\{v',v''\}$ be the pairs of points of X_n that give us the double points of χX_n. Then $K_{X_n}+\delta-u'-u''$ and $K_{X_n}+\delta-v'-v''$ move in two linear systems of dimension 2 and we can write

$$K_{X_n}+\delta-u'-u'' \equiv K_{X_n}-t \quad \text{and} \quad K_{X_n}+\delta-v'-v'' \equiv K_{X_n}-t'$$

for some elements t and t' of X_n. So

$$\delta \equiv u'+u''-t \equiv v'+v''-t'$$

and

$$\xi \equiv K_{X_n}-g_0+u'+u''-t \equiv K_{X_n}-g_0+v'+v''-t'$$

As ξ is a g^1_3, from this we first deduce that $K_{X_n}-g_0 \equiv t'+t$ and then that

$$\xi \equiv u'+u''+t' \equiv v'+v''+t$$

Let p' and p'' be the points of X_n that we identify to obtain X. Write $\xi' = K_{X_n}-\xi$, then $K_{X_n}+\delta-p'-p'' \equiv K_{X_n}+\delta-2\delta \equiv K_{X_n}-\delta \equiv K_{X_n}-u'-u''+t \equiv \xi+\xi'-u'-u''+t \equiv \xi'+t'+t$.

So, in particular, $K_{X_n}+\delta-p'-p''-t'-t$ is a g^1_3 and the points p', p'', t, t' are on a line in $\mathbb{P}T_0A$. So this line is in q and we have $p'+p''+t'+t \in g_0$. It follows that

$$\xi \equiv K_{X_n}+\delta-g_0 \equiv K_{X_n}+\delta-p'-p''-t'-t \equiv \xi'$$

So X_n is a curve of genus 4 with a vanishing theta-null. We also deduce that $2g_0 \equiv K_X+\Delta$, so as $h^0(g_0-p'-p'') > 0$, g_0 induces a vanishing theta-null on X. Also notice that the pencil of planes through a quadrisecant line to χX_n cuts, residually on X, a *singular* vanishing theta-null. Moreover, $h^0(X,\omega_X\otimes\eta(g_0)) = 2 = h^0(X,\eta(g_0))$, so by [B1],

$$A \in \theta_{null}$$

The locus of the abelian varieties that have a two-dimensional family of these Prym-curves has dimension less than or equal to 6.

Suppose now that χX_n is contained in a singular quadric q. We first notice that for degree reasons χX_n has to pass through the singular point of q and the ruling of q cuts a g^1_3 on q. Suppose that we do have a two-dimensional family of curves with one double point in $P^{-1}(A)$.

The abelian variety $J\widetilde{X}_n$ is isogenous to the product $JX_n \times A$ by an isogeny of bounded degree. So $\widetilde{X}_n$ and hence the ramification points of $\widetilde{X}_n \to X_n$ are determined by X_n and A . We deduce that the curves X_n describe at least a curve in $\mathcal{M}_4$. Equivalently, their jacobians describe at least a curve in $\mathcal{J}_4$ (or $\overline{\mathcal{J}}_4$) . As $\mathrm{Pic}\mathcal{A}_4 = \mathbb{Z}$ ([SV]), the divisor θ_{null} is ample and intersects this (complete) curve or surface at (generalized) jacobians of curves X'_n of arithmetic genus 4 with a vanishing theta-null. Notice that for a generic X_n the g^1_3 cut on X_n by the ruling of q verifies $2g^1_3 \equiv K_{X_n} + \delta - t$ where δ is the divisor class associated to the ramified cover $\widetilde{X}_n \to X_n$ and t is the vertex of q . The unique g^1_3 on a (fixed) X'_n is the specialization of the (two) g^1_3's on the generic X_n's . So this is also true for the g^1_3 of X'_n . In this case we have in addition that $2g^1_3 \equiv K_{X'_n}$, but this implies that $K_{X'_n} + \delta$ has a base point and $\chi X'_n$ has degree 6 .

We deduce that the closure of $\mathcal{P}_{6,1}$ surjects onto $\mathcal{A}_4$ with fibers of dimension 1 *everywhere* on $\mathcal{P}_{6,1}$.

If $s = 2$, for generic X , χX_n is not contained in a quadric. Hence P_2 is generically étale and the closure of $\mathcal{P}_{6,2}$ surjects onto $\mathcal{A}_4$. As χX_n is of degree 6 , there is at most one quadric containing it, so that the fibers of P_2 have dimension at most 1 .

If $s = 3$, the dimension of $H^1(X, Hom(\Omega_X, \mathcal{O}_X))$ is 9 . For all X , there is exactly one quadric containing χX_n . Hence P_3 is *everywhere* étale, in particular, the closure of its image is a divisor in $\mathcal{A}_4$.

If $s = 4$, the dimension of $H^1(X, Hom(\Omega_X, \mathcal{O}_X))$ is 8 . For all X , there is exactly one pencil of quadrics containing χX_n . Hence P_4 is also everywhere étale and the closure of its image has codimension 2 in $\mathcal{A}_4$.

If $s = 5$, the dimension of $H^1(X, Hom(\Omega_X, \mathcal{O}_X))$ is 7 . Then χX_n is a twisted and there is exactly a net of quadrics containing it. So P_5 is everywhere étale and the closure of its image has codimension 3 in $\mathcal{A}_4$.

(7.3) Now suppose that A is isogenous to a product of lower dimensional ppav's . Then $P^{-1}(A)$ also contains reducible curves. The curve X has at most four components. As in [B1] we associate a graph to X : the vertices of the graph are the irreducible components of X and each edge between two vertices represents a point of intersection of the two irreducible components. Each irreducible component has to intersect the rest of the curve in at least four points because otherwise A would be a product [B1]. We investigate the various possibilities for the graphs :

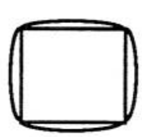
- We first see that there are at most two curves with four components in $P^{-1}(A)$:

In this case the genus of X is greater than or equal to 12 - 3 = 9

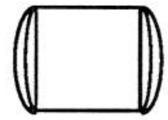
In this case $g \geq 11 - 3 = 8$

$g \geq 10 - 3 = 7$

$g \geq 9 - 3 = 6$



$g \geq 8 - 3 = 5$: this is a possible case but then all the components have to be of genus 0 , their double covers will be elliptic curves. Elliptic curves are rigid in an abelian variety hence there is only one curve of this form in $P^{-1}(A)$.



$g \geq 5$: this case is analogous to the preceding case.

- Suppose now that X has three components. Notice that we have to minimize the number of edges because they increase the genus. The possibilities are :

$g \geq 8 - 2 = 6$

$g \geq 7 - 2 = 5$: X has three rational components, $\tilde{X}$ has two elliptic components and one genus 2 component. The elliptic components embed in A and do not move. The jacobian of the genus 2 component also embeds in A and does not move either because it is an abelian subvariety of A . So there is only one such Prym-curve in $P^{-1}(A)$.

$g \geq 6 - 2 = 4$: here we have two components C_1 , C_2 of genus 0 and one component C_0 of genus 1 . We are going to show that there is at most a one-dimensional family of such Prym-curves in $P^{-1}(A)$.

As each component C_i intersects the rest of the curve in four points that are the ramification points of the cover $\tilde{C}_i \to C_i$, $\tilde{X}$ is the union of two elliptic curves and a curve of genus 3 . The elliptic components embed in A and are rigid in A . The Prym variety of the ramified cover $\tilde{C}_0 \to C_0$ embeds in A and is the jacobian of a curve of genus 2 . As the jacobian of the curve of genus 2 is an abelian subvariety of A , the curve of genus 2 does not move either. We claim that there is exactly a one-dimensional family of such curves in $P^{-1}(A)$. For this it will be enough to show that there is exactly a one-dimensional family of ramified covers $\tilde{C}_0 \to C_0$ with Prym variety a fixed ppav of dimension 2. Let δ be the divisor class on C_0 such that

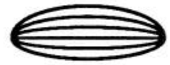
$$\tilde{C}_0 = \mathrm{Spec}_{C_0}(\mathcal{O}_{C_0} \oplus \mathcal{O}_{C_0}(\delta))$$

then the Prym-canonical embedding of C_0 is given by $\omega_{C_0}(\delta)$. This is of degree 2 and is onto from C_0 to $\mathbb{P}^1 = \mathbb{P}T_0P(\tilde{C}_0, C_0)$. The codifferential of the Prym map from the space of curves of genus 1 with a cover ramified at four points to the space of curves of genus 2 is given by multiplication

$$S^2H^0(C_0, \omega_{C_0}(\delta)) \to H^0(C_0, (\omega_{C_0}(\delta))^2)$$

this is injective because the Prym-canonical image of C_0 is not contained in any quadric. Hence the fibers have dimension 1 everywhere as required.

- Suppose now that X has two components. We have the following possibilities :



$g \geq 6 - 1 = 5$: all the components are rational. There is only one such Prym-curve in $P^{-1}(A)$.

$g \geq 4 - 1 = 3$. In this case either X is the union of a rational curve and a curve of genus 2 or X is the union of two elliptic curves.

In the first case, $\tilde{X}$ is the union of a curve of genus 5 and an elliptic curve meeting in four points. The fibers of the restriction of P to this locus have dimension 1 : Arguing as in the last case where X has three components we have to show that, if C_0 is the component of genus 2 of X, the map

$$S^2H^0(C_0, \omega_{C_0}(\delta)) \to H^0(C_0, (\omega_{C_0}(\delta))^2)$$

is injective. The image of C_0 in $\mathbb{P}^2 = \mathbb{P}T_0P(\tilde{C}_0, C_0)$ is a quartic curve. If this quartic curve is twice a conic, then $\omega_{C_0}(\delta)$ is equal to $3g^1_2$ and δ is equal to the unique g^1_2 of C_0. So the four points of ramification t_i $(1 \leq i \leq 4)$ of $\tilde{C}_0 \to C_0$ verify (for instance) t_i

$+ t_{i+1} \in g_2^1$. The moduli space $\mathcal{S}_1$ of such curves X has dimension 6 . Also, the moduli space $\mathcal{S}_2$ of curves corresponding to the last graph in the case of three components has dimension 6 . It is immediately seen that the intersection $\mathcal{S}_1 \cap \mathcal{S}_2$ has dimension 5 . We saw that the fibers of the restriction of P to $\mathcal{S}_2$ have dimension 1 . Suppose that at A the intersection $P^{-1}(A) \cap \mathcal{S}_1$ has dimension 2 .

The jacobian of $\tilde{C}_0$ is isogenous to the product of C_0 and $P(\tilde{C}_0,C_0)$ by an isogeny of fixed degree. So, if C_0 and $P(\tilde{C}_0,C_0)$ are fixed, C_0 is fixed ; hence the ramification points of $\tilde{C}_0 \to C_0$ are fixed. We deduce that the curves C_0 describe a family of dimension 2 in $\mathcal{M}_2$. Hence their jacobians describe a divisor in $\mathcal{A}_2$ (or $\overline{\mathcal{A}}_2$). This must be ample because $\mathrm{Pic}\mathcal{A}_2 \cong \mathbb{Z}/10\mathbb{Z}$ (see [H]) so it encounters the boundary $\overline{\mathcal{A}}_2 \setminus \mathcal{A}_2$. This implies that $P^{-1}(A)$ intersects $\mathcal{S}_2$ and A is in $P(\mathcal{S}_2) \subset \mathcal{A}_{n11}$.

In the second case $\tilde{X}$ is the union of two curves of genus 3 meeting in four points. Arguing as before we that the fibers of the restriction of P to this locus have dimension 2. Q.E.D.

(7.4) *Remark* : It is proven in [B1] (p. 183) that a reducible curve of genus 5 with two elliptic components meeting in four points is a limit of smooth curves of genus 5 with a vanishing theta-null.

(7.5) **PROPOSITION** : *If* A *is not in* $\overline{\mathcal{J}}_4 \cup \mathcal{A}_{n11}$ *, then the only base point of* $|2\Theta|_{00}$ *is* 0 .

Proof : By 7.1, a generic curve in $P^{-1}(A)$ is smooth (and irreducible).

The map $\Sigma_A = (\cup\, \Sigma(X)) \to A$ is surjective for A generic, hence $\Xi \to \mathcal{K}$ or $\mathbb{A}$ is surjective (because the Prym map is proper and each $\Sigma(X)$ is complete), hence $\Sigma_A \to A$ is surjective for all A .

As $\tilde{h}$ is generically finite and $\Sigma_A \to A$ is surjective, for X generic in a component of $P^{-1}(A)$, the image of $\Sigma(X)$ by $\tilde{h}$ is two-dimensional. This is actually a plane because it is so for A generic. Hence there is *exactly* 1 pencil l_X of Γ_{00}-divisors containing $\Sigma(X)$. The base locus of l_X will be at most the union of $\Sigma(X)$ and a divisor which does not contain $\Sigma(X)$. Notice that, for homology class reasons, the map $X \to l_X$ is generically finite.

If the dimension of $V(\Gamma_{00})$ is ≥ 2, then $\Sigma(X)$ will intersect the base locus because the homology class of $\Sigma(X)$ is $2[\Theta]^2$. Also $\Sigma(X)$ cannot be contained in the base locus by the surjectivity of $\Sigma_A \to A$.

By the corresponding statement for A generic and because a generic curve in $P^{-1}(A)$ is smooth, every element of $|2\Theta|_{00} \setminus l_X$ cuts on $\tilde{X}^{(2)}$ the inverse image of the set of pairs in $X^{(2)}$ which correspond to bisecants in a quadric containing κX (union the set of $\{p,\sigma p\}$). However the intersection of the quadrics containing κX does not contain any bisecant of κX unless X is trigonal, in which case $P(\tilde{X},X)$ is in the closure of $\mathfrak{J}_4$.

Suppose now that $V(\Gamma_{00})$ has dimension less than 2.

Suppose that the base locus of every pencil l_X (which we defined for X generic in a component F_0 of $P^{-1}(A)$) contains a divisor D_0. Then, as $\tilde{h}$ is generically finite, D_0 is the same for every pencil l_X. If the lines l_X span a hyperplane, then every generic Γ_{00}-divisor intersects D_0 in the same surface : $V(\Gamma_{00})$ will be two-dimensional. So the lines l_X are all in a plane : the pencil of hyperplanes containing this plane will be a line in $(|2\Theta|_{00})^*$ contained in every plane $h(\Sigma(X))$. Looking at the inverse image of this line, we see that for every $X, Y \in F_0$, $\Sigma(X) \cap \Sigma(Y)$ is the same curve. Take X and Y tetragonally related and smooth and consider $\tilde{X}^{(2)}$ and $\tilde{Y}^{(2)}$ instead. Fix X and let Y vary. Then as above we can use the smoothness assumption and the generic result (5.11) to determine the inverse image of $\Sigma(X) \cap \Sigma(Y)$ in $\tilde{X}^{(2)}$. We conclude that as X is nontrigonal, two of these cannot have common components (other than the set $\Delta' = \{\{p,\sigma p\} : p \in \tilde{X}\}$).

So the base locus of l_X is two-dimensional. Notice that by 2.4, X_λ is defined for smooth curves in $P^{-1}(A)$. As l_X is the limit of generic pencils l_X and containment is a closed condition we deduce that the base locus of 1_X is be the union of $\Sigma(X)$ and $\Sigma(X_\lambda)$. Also, the X_λ's describe a two-dimensional family because, as we saw in 2.4, if X is tetragonally related to Y through g_4^1 on Y, then X_λ is tetragonally related to Y through $K_X - g_4^1$; hence if a one-dimensional family of X's have the same X_λ, then X_λ will have a two-dimensional family of g_4^1's and by [B1] X_λ would be hyperelliptic. Hence a generic X_λ is smooth.

Any base point of $|2\Theta|_{00}$ will be either in $\Sigma(X)$ or in $\Sigma(X_\lambda)$. This is taken care of as above. Q.E.D.

(7.6) **COROLLARY** : *If* A *is not in* $\overline{\mathfrak{J}}_4 \cup \mathfrak{A}_{n11}$ *, then the base locus of* $|2\Theta|_{00}$ *is* 0 *with multiplicity* 2^8 .

Proof: As the generic curve in $P^{-1}(A)$ is smooth, the proof of this is exactly like that of 5.15 and 5.22. Q.E.D.

As in 5.23 we have :

(7.7) **COROLLARY** : *If* A *is not in* $\overline{\mathfrak{J}}_4 \cup \mathfrak{A}_{n11}$ *, the base locus of* $\tau|2\Theta|_{00}$ *is empty.*

(7.8) **COROLLARY** : *If* A *is not in* $\overline{\mathfrak{J}}_4 \cup \mathfrak{A}_{n11}$ *, the linear system* $\tau|2\Theta|_{00}$ *has projective dimension greater than or equal to* 3 *. Equivalently, the Kernel of* $\tau : \Gamma_{00} \to S^4$ *has dimension (as a vector space) at most* 1 .

Proof : If the linear system is a net (or of smaller dimension), its base locus will be nonempty.

8. CUBIC THREEFOLDS ON $\mathcal{A}_4 \setminus (\overline{\mathcal{J}}_4 \cup \mathcal{A}_{n11})$

We wish to prove what has been announced in 1.8 . Using the results of section 7, we first define the cubic threefold for every $A \in \mathcal{A}_4 \setminus (\overline{\mathcal{J}}_4 \cup \mathcal{A}_{n11})$.

(8.1) **LEMMA** : *Let* A *be in* $\mathcal{A}_4 \setminus (\overline{\mathcal{J}}_4 \cup \mathcal{A}_{n11})$. *Then* $P^{-1}(A)$ *is two-dimensional*.

Proof : The generic Prym-curve in $P^{-1}(A)$ is smooth. If the fiber of the Prym map is three-dimensional (or more), then there will be (at least) a two-dimensional family of singular Prym-curves in $P^{-1}(A)$:

The set of singular curves is a divisor and it intersects every $P^{-1}(A)$ in a curve by 7.1 . At a generic intersection point the intersection V of the Zariski tangent spaces to $P^{-1}(A)$ and the divisor of singular curves will have dimension 2 or 3 . As we are supposing that $P^{-1}(A)$ has dimension ≥ 3 , (at least) a two-dimensional subspace of the infinitesimal deformations in V will correspond to global deformations.

This is ruled out in 7.1 . Q.E.D.

Recall that in the proof of 7.5 we defined l_X for X greneric in a component of $P^{-1}(A)$ and showed that its base locus is two-dimensional. Also recall that X_λ was defined in 2.4 for all smooth (nonhyperelliptic, but this is always true because of the hypothesis on A) curves X and we saw in the proof of 7.5 that for X generic in a component of $P^{-1}(A)$ the base locus of l_X is the union of $\Sigma(X)$ and $\Sigma(X_\lambda)$.

(8.2) **PROPOSITION** : *Let* A *be in* $\mathcal{A}_4 \setminus (\overline{\mathcal{J}}_4 \cup \mathcal{A}_{n11})$. *Then the pencil* l_X *is defined for all* $X \in P^{-1}(A)$.

Proof : The image of $\Sigma(X)$ by $\tilde{h}$ is contained in a plane for all X because it is so generically. Also, $\tilde{h}(\Sigma(X))$ is not a line because it contains the image under $\tilde{h}$ of χX .

By the analysis in 7.1 and 7.8, $\tilde{h}(\chi X)$ cannot be a line or several times a line. Q.E.D.

DEFINITION : For A in $\mathcal{A}_4 \setminus (\overline{\mathcal{J}}_4 \cup \mathcal{A}_{n11})$ define T to be the smallest variety containing all the lines l_X for $X \in P^{-1}(A)$.

As the image of $\Sigma(X)$ in $|2\Theta|_{00}$ spans a plane, no divisor in $|2\Theta|_{00} \setminus l_X$ contains $\Sigma(X)$. As in the proof of 7.5, for smooth X, we still have the projection from l_X : $|2\Theta|_{00} \to N$. The lines corresponding to curves tetragonally related to X go to points on the plane quintic Q. Notice that as we have a two-dimensional family of lines l_X, a generic hyperplane section of T contains a finite number of these lines. From the generic case we deduce that the number of distinct lines l_X in a generic hyperplane is less than or equal to 27. We have :

(8.3) **LEMMA** : *A generic hyperplane section of* T *contains exactly* 27 *lines* l_X.

Proof: Let H be a generic hyperplane in $|2\Theta|_{00}$ and suppose that H contains less than 27 lines l_X. Then there is a line $l = l_Y$ (for a smooth Y because H is generic) in H which counts twice.

We can find a one-parameter family $\{A_t\}$ of ppav's with central fiber $A_0 = A$ such that the generic member of $\{A_t\}$ is a generic ppav. For such a family we can find a family of generic hyperplanes H_t with $H_0 = H$ and a family of pairs of lines $\{l_{1t}, l_{2t}\}$ in $T_t \cap H_t$ such that for $t \neq 0$, l_{1t} and l_{2t} are distinct and $l_{10} = l_{20} = l$. Let $l_{1t} = l_{U_t}$, $l_{2t} = l_{Y_t}$ and let $l_{3t} = l_{X_t}$ be a line incident to both l_{1t} and l_{2t} for all t. If l_{1t} and l_{2t} meet for t generic suppose in addition that l_{3t} doe not pass through the intersection point of l_{1t} and l_{2t}.

i) First suppose that the lines l_{1t} and l_{2t} do not meet for $t \neq 0$. Then they span H_t for each $t \neq 0$. If the three lines l_{1t}, l_{2t}, l_{3t} come together when t goes to 0, replace l_{2t} by l_{3t} and reason as in ii) below.

Otherwise, when l_{1t} and l_{2t} come together, the two g^1_4's relating X_t to Y_t and U_t come together or become opposite because the intersection of $\Sigma(Y)$ with $\Sigma(X)$

is the set of pairs projecting to pairs $\{p,q\} \in X^{(2)}$ such that $h^0(g^1_4 - p - q) > 0$ (reason on $\tilde{X}^{(2)}$ and $\tilde{Y}^{(2)}$ as in the proof of 7.5). Hence, as in the proof of 5.20, the Del-Pezzo surface $q_{1t} \cap q_{2t}$ acquires a double point. Here q_{1t} and q_{2t} are the quadrics of rank 4 in the canonical space of X_t associated to the two g^1_4's . However the generic quartic Del-Pezzo containing κX is smooth. (We are using the fact that a generic Prym-curve in $P^{-1}(A)$ is nonsingular.)

ii) Now suppose that for all $t \neq 0$, l_{1t} and l_{2t} meet. Then by 5.12 we can suppose that X_t , Y_t and U_t are tetragonally related. At the limit, $X = X_0$, $Y = Y_0$ and $U = U_0$ are smooth as H is generic and they are also tetragonally related. However $l_Y = l_U = 1$ so we have three possibilities :

a) $Y = U$. For any tetragonal triple there is a trigonal curve V of genus 6 trigonally related to all three curves and the three points of order 2 on JV for the three trigonal constructions are the three nonzero elements of a totally isotropic rank 2 vector space in $(JV)_2$. For each g^1_4 on Y there is exactly one pair {trigonal curve, point of order 2 } trigonally related to Y through that g^1_4 . So the points of order 2 in JV corresponding to $Y = U$ must coincide. Then the sum of these points of order 2 is equal to the point of order 2 corresponding to X . So the point of order 2 corresponding to X is 0 : this is not possible because then V would be isomorphic to X and the g^1_4 on X will have a base point.

b) $Y = Y_\lambda$ and $U = U_\lambda$. By a) we can suppose that $Y \neq X$. So, as Y determines its g^1_4 as in i), this g^1_4 is a theta-null on X and this is not the case for a generic g^1_4 .

c) $Y = U_\lambda$. As in a) $Y \neq X$ and $U \neq X$. The curve X is tetragonally related to U_λ through $K_X - g^1_4$: we conclude as in b) . Q.E.D.

(8.4) Let $\tilde{\mathbb{A}}$ be the blow up of the universal family $\mathbb{A} \to \mathcal{H}$ along the zero section. Let $\tilde{\mathbb{A}}'$ be its restriction to the locus $\mathcal{A}_4 \setminus (\bar{\mathcal{J}}_4 \cup \mathcal{A}_{n11})$. By section 7 , the Γ_{00} maps define a *morphism*

$$\tilde{\mathbb{A}}' \to \mathbb{P}^*|_{\mathcal{a}_4 \setminus \bar{\mathcal{J}}_4 \cup \mathcal{a}_{p11}}$$

which is generically finite on each A and the branch locus of this morphism intersects each fiber $(|2\Theta|_{00})^*$ in a divisor. One component of the branch locus is the image of the exceptional locus $\mathbb{E}$ in $\tilde{\mathbb{A}}'$. From the generic case we deduce that the second component of the branch locus is the set of hyperplanes which contain less than 27 lines.

If T is a union of planes, then by 8.3, it will have to be a union of 27 distinct planes. Then the set of hyperplanes containing less than 27 lines will be exactly the set of hyperplanes containing one of the planes of T : this is impossible by the above discussion.

So T is a hypersurface and by continuity the degree of T is less than or equal to 3 . As the plane quintics Q are reduced we have that T is neither a hyperplane nor a union of hyperplanes. So T can at worst degenerate to a quadric or the union of a quadric and a hyperplane.

(8.5) **LEMMA** : T *is an irreducible cubic threefold.*

Proof: By 8.3, a generic hyperplane section of T contains exactly 27 *distinct* lines l_X. These must intersect at least as much as the 27 lines in a generic hyperplane section of the cubic threefold associated to a generic abelian variey.

Consider a generic line l_X and consider the family of pairs of lines corresponding to pairs of curves tetragonally related to X through the same g^1_4. Generically, by 8.3, each of these is a pair of distinct lines, distinct from l_X and together with l_X it gives a triple of distinct lines in a plane section of T . Again by 8.3, a generic such plane section of T does not contain a one-dimensional family of lines l_X and contains exactly three distinct lines. We first deduce from this that T is not a quadric because otherwise each of these planes would be contained in T and as there is at most a one-dimensional family of planes in a nondegenerate quadric each plane would have to contain a one-dimensional family of lines l_X.

Suppose now that T is the union of a quadric and a hyperplane H_0. Choose a generic hyperplane H. Out of each triple of lines in a plane in H, one has to be in H_0. Then $H \cap H_0$ would be a plane containing more than 3 lines l_X, so it contains a one-dimensional family of them. However this cannot be true for a generic hyperplane H because otherwise $P^{-1}(A)$ would be more than two-dimensional. Q.E.D.

(8.6) **PROPOSITION** : T *is a cubic threefold with at most isolated double points as singularities.*

Proof : Suppose that the cubic threefold T has a triple point t. Then T is the cone of vertex t over an irreducible cubic surface S. The family of lines in T has the following components : lines through t and lines in the planes projecting to lines in S. A generic hyperplane section of T is isomorphic to S. So by 8.3, S is smooth. Then the lines l_X will be in the planes above the lines in S : T would be a union of planes.

Suppose now that T has a double curve C. Then every hyperplane section of T will be singular. Also, as T is irreducible, a general hyperplane section of T will contain a finite number of lines and as it is singular it will contain less than 27 lines. This is ruled out by 8.3. Q.E.D.

From 8.6 we deduce in particular that, as in the generic case, T has a structure of conic bundle over $\mathbb{P}^2$ with discriminant curve Q for each Prym-curve X. We have

(8.7) **THEOREM** : T *is singular if and only if* $A \in \mathcal{A}_4 \setminus (\overline{\mathcal{J}}_4 \cup \mathcal{A}_{n11})$ *has a vanishing theta-null.*

For the proof we first need

(8.8) **LEMMA** : *If* X *is smooth, then a vanishing theta-null* E *on* X *such that* $h^0(\eta(E))$ *is odd corresponds to a double point on* X_λ.

Proof : Parametrize A with X. Let $\pi : \tilde{X} \to X$ be the projection. Recall that $\tilde{X}_\lambda$ is the set of embeddings of $\tilde{X}$ in Θ, i.e., $\tilde{X}_\lambda$ can be identified with

$$\{ E' \in Pic^8\tilde{X} : \pi_* E' \equiv K_X \ , h^0(E') \text{ is odd} \} .$$

Let $E' = \pi^*E$, then E' verifies

$$\pi_*E' \equiv 2E \equiv K_X \quad , \quad h^0(E') = h^0(E) + h^0(\eta(E)) \quad , \quad E' = K_{\tilde{X}} - E' = \sigma E'$$

As $h^0(E)$ is even, we see that $h^0(E')$ is odd. So E' is an element of $\tilde{X}_\lambda$. The covering involution σ_λ on $\tilde{X}_\lambda$ is induced by the involution $E' \mapsto K_{\tilde{X}} - E'$ on $Pic^8\tilde{X}$. So the cover $\pi_\lambda : \tilde{X}_\lambda \to X_\lambda$ is ramified at E' . So X_λ is singular.

Proof of theorem 8.7 : T is singular if and only if, for all X , Q is singular. Equivalently, for all X , X has a vanishing theta-null because the generic curve in $P^{-1}(A)$ is smooth by 7.1 . Hence the locus of singular cubic threefolds is contained in the image by P of the one vanishing theta-null locus $\theta_{null,5}$ in $\mathcal{P}_5 \setminus P^{-1}(\overline{\mathcal{J}}_4 \cup \mathcal{A}_{n11})$. This has dimension 11 .

By [B1] , $\theta_{null,5}$ has exactly two irreducible components θ_{null1} and θ_{null2} . These are described as follows. Let $X \in \theta_{null,5}$ be generic. Let g^1_4 verify $2g^1_4 = K_X$. Then

- $X \in \theta_{null1}$ if and only if $h^0(\pi^*g^1_4)$ is even,
- $X \in \theta_{null2}$ if and only if $h^0(\pi^*g^1_4)$ is odd.

By [B1] , $P^{-1}(\theta_{null})$ is the closure of θ_{null1} and θ_{null} is the closure of $P(\theta_{null1})$. By 8.8 and 7.1, θ_{null2} maps *onto* $\mathcal{A}_4 \setminus (\overline{\mathcal{J}}_4 \cup \mathcal{A}_{n11})$ with one-dimensional fibers everywhere.

It follows that T is singular on θ_{null} only. Q.E.D.

9. (SINGULAR) CUBIC THREEFOLDS OVER $\mathfrak{J}_4$

We wish to define the cubic threefold for the jacobian of a smooth nonhyperelliptic curve of genus 4.

Let $A = JC$ be the jacobian of a nonhyperelliptic curve. By [B1], $P^{-1}(A)$ has two components interchanged by λ, namely, those described in 4.2. In particular, $P^{-1}(A)$ is two-dimensional.

By [W3], the base locus $V(\Gamma_{00})$ of $|2\Theta|_{00}$ is equal to $C\text{-}C \cup \{\pm(K_C-2g^1_3)\}$.

Using 4.3 it is easily seen that

$$\Sigma(C_{pq}) \;=\; C\text{-}C \cup W_2\text{-}p\text{-}q \cup p\text{+}q\text{-}W_2 \;.$$

Also, for a trigonal $X \in P^{-1}(A)$, associated to g^1_4 on C:

$$\Sigma(X) \;=\; \{s+t-s'-t' : h^0(g^1_4 - s - t) > 0 \text{ and } h^0(g^1_4 - s' - t') > 0\} \;.$$

From this we deduce that we can define the pencil $l_X = l_{X_\lambda}$ as in 7.5. As in the proof of 7.5, the base locus of each pencil will be two-dimensional because otherwise the base locus of $|2\Theta|_{00}$ will be a divisor. So we can define T to be the reduced subvariety of $|2\Theta|_{00}$ containing the lines corresponding to the Prym-curves for A (T is irreducible because $P^{-1}(A)$ has two irreducible components exchanged by λ).

We still have the projection from l_X : $|2\Theta|_{00} \to N$. (because the image of $\Sigma(X)$ (resp. $\Sigma(C_{pq}) \setminus C\text{-}C$) by h is a plane for a trigonal X (resp. singular C_{pq})). Under this projection, the lines corresponding to Prym-curves tetragonally related to X go to points on the plane quintic Q parametrizing singular quadrics through the canonical model of X. So T cannot be a plane neither a hyperplane. T is a threefold and by continuity the degree of T is less than or equal to 3.

Recall that in 4.5 we computed the number of lines in a generic intersection of translates of Θ for a jacobian. The computation works for the jacobian of a smooth nonhyperelliptic curve. Hence a generic hyperplane contains exactly 27 distinct lines. These intersect at least as much as in the case where A is generic.

Then, using the incidence configuration of the lines in a generic hyperplane, one sees that if T is a quadric, then the lines cannot be all distinct. Also the lines cannot be distinct unless the hyperplane section of T is a smooth cubic surface.

Also, all the lines cannot pass through the same point by the surjectivity of $\Sigma_A \to A$. So T has no triple point.

We have proved :

(9.1) **PROPOSITION** : T *is a cubic threefold with at worst a finite number of double points.*

From the above it also follows that T has conic-bundle structures over the nets N with discriminant curves Q. As X (generic in $P^{-1}(A)$) is smooth and trigonal, Q is isomorphic to X with two points identified and has exactly one double point. It follows that

(9.2) **PROPOSITION** : T *has exactly one double point. In particular, every line in* T *is of the form* l_X *for some Prym-curve* X.

We still have the double cover $l : \tilde{F} = P^{-1}(A) \to F$.

Recall that the two components of $P^{-1}(A)$ are each isomorphic to $C^{(2)}$. Let ξ be one of the two (possibly equal) g^1_3's on C and let $\xi' = K_C - \xi$. The two components intersect in the set of singular curves C_{pq} such that $h^0(\xi - p - q) > 0$ or $h^0(\xi' - p - q) > 0$. Suppose, for instance, that $h^0(\xi - p - q) > 0$. Then ξ induces a g^1_3 on C_{pq}. Hence, by continuity (from the smooth trigonal case), the plane quintic Q_{pq} associated to C_{pq} is isomorphic to C_{pq} with two points identified. However Q_{pq} is also isomorphic to the curve C' parametrizing the set of lines through the double point of T with two pairs of points identified. Hence $C' = C$:

(9.3) **LEMMA** : C *parametrizes the set of lines through the double point of* T .

(9.4) **COROLLARY** : *The quadric tangent cone to* T *at its double point has rank* ≥ 3 .

Proof : This is because, by continuity from the case where T has an ordinary double point, the embedding of C in the exceptional $\mathbb{P}^3$ above the double point of T is the canonical embedding of C.

(9.5) *Remark* : If we had known that T has an *ordinary* double point, we could have deduced the above from the fact that $F = C^{(2)}$ and from the results of [CG].

10. DOUBLE SOLIDS OVER $\mathcal{J}_4$ AND θ_{null}

Generically over $\mathcal{J}_4$ and θ_{null} , because the generic plane quintics Q have one ordinary double point, T has exactly one *ordinary* double point.

Consider the map $E : \mathcal{Z}_A \to K(A) = A_{/\pm 1}$ which to each double solid Z associates the element $\bar{a}$ of K(A) ($a \in A$) with the property $\Theta.\Theta_a = E_Z$ the Fano variety of lines in $\tilde{Z}$. For a generic ppav, by 3.5, the map E is of degree 16 onto the image of $\mathcal{E}$ in K(A). The image of $\mathcal{E}$ by $\tilde{h}$ is the set of hyperplanes of $|2\Theta|_{00}$ which contain less than 27 lines. Equivalently, the image of $\mathcal{Z}_A$ by the composition $D^* = \tilde{h}.E$ is T^*. As T is smooth, it is birationally equivalent to T^* ([CG]). The composition of D^* with the birational equivalence $T^* \to T$ is just $D : Z \mapsto D_Z \in |2\Theta|_{00}$.

Generically on $\mathcal{J}_4 \cup \theta_{null}$, the set of hyperplanes in $|2\Theta|_{00}$ which contain less than 27 lines is equal to $T^* \cup H_0$ where $H_0 \cong \mathbb{P}^3$ is the set of hyperplanes in $|2\Theta|_{00}$ through the double point t of T. If A is in θ_{null} , put the double point of Θ at the origin. We have

(10.1) **PROPOSITION** : *The pullback of* H_0 *is the divisor of zeros of* θ^2 *in the one vanishing theta-null case and* $\theta_\xi.\theta_{-\xi}$ *in the jacobian case.*

(Recall that θ is a nonzero section of $\mathcal{O}_A(\Theta)$ and θ_ξ is a nonzero section of $\mathcal{O}_A(W_3 - \xi)$ in the curve case, ξ being a g^1_3 on the curve, W_3 the variety of effective divisor classes in $\mathrm{Pic}^3 C$.)

Proof: On θ_{null} we know that $T^* \cup H_0$ is contained in the branch locus of $\tilde{h}$ because it is the specialization of the generic T^*.

Generically on $\mathcal{J}_4$ consider a generic singular Prym-curve C_{pq} . Consider a generic pencil of hyperplanes in $|2\Theta|_{00}$ containing the line associated to C_{pq} and with special member a hyperplane tangent to T at a unique point (or a generic hyperplane through the double point of T) which is not on the line of C_{pq} . Each of these hyperplanes projects to a line in the net N of quadrics containing the canonical model of C_{pq} . We see that the special hyperplane projects to a line l which is simply tangent to the plane quintic Q in N (or is a generic line through the double point of Q). Hence the Del-Pezzo surface which is the intersection of the quadrics in l has one node and as in the proof of 5.15 we see that the number of preimages by $\tilde{h}$ of the special member of the pencil is less than the number of preimages of a general member. So we see that also on $\mathfrak{J}_4$, $T^* \cup H_0$ is contained in the branch locus of $\tilde{h}$.

So the inverse image of H_0 in A is split. Now the proposition follows from the fact that the Picard group of JC has rank 1 for JC generic in $\theta_{null} \cap \mathfrak{J}_4$. Q.E.D.

We are first going to determine the double solids (or degenerations of double solids) with intermediate jacobian a generic element of θ_{null} .

By [C1], when the equation of the branch locus B of Z is a second degree homogeneous form in the second degree forms vanishing on the double points p_i of Z , then JZ has a vanishing theta-null. The moduli space of these double solids has dimension 12 , hence the generic element of θ_{null} is obtained in this way. Also $B \rightarrow \rho(B)$ is of degree 2 and $\rho(B) = 2q_0$ where q_0 is a quadric ([C1]). Also, by [C1], the Prym-curves associated to Z have *two* vanishing theta-nulls. So their plane quintics have two double points. As T is generic with one ordinary double point it follows that the set $(\mathcal{Z}_A)_0 \subset \mathcal{Z}_A$ of these double solids is blown down to the singular point of T by the map D . Hence

(10.2) **PROPOSITION :** $D^*((\mathcal{Z}_A)_0) = H_0$.

Conversely, it is immediately seen that, for a double solid Z , $\rho(B) = 2q_0$ only if B is a quadric in the quadrics through the p_i's .

(10.3) **COROLLARY** : q_0 *is the tangent cone to* Θ *at its (unique because* A *is a generic element of* θ_{null} *) double point.*

Proof : Use 10.1 and 10.2 .

Let $X \in P^{-1}(A)$ be generic : X is a generic element of θ_{null1} . Let $g = g_4^1$ be the unique theta-null of X . Let $\{p,q\} \in X^{(2)}$ be a pair such that the image X^{pq} of X in $\mathbb{P}^2$ given by $g_6^2 = K_X - p - q$ has an everywhere tangent conic and is otherwise generic. Then (2.2) there a unique double solid Z such that $\tilde{X}$ parametrizes the lines in Z through p_1 . By adjunction, the sums $D_1{+}D_2$, where $D_1 \in g$ and $D_1 \in K_X - g$, are cut on X^{pq} by a conic (= $\mathbb{P}^1$) of cubics through $p_2,\ldots,p_6$. As $g = K_X - g$, all these cubics are everywhere tangent to X^{pq} . Conversely any such family of cubics cuts, residually on X^{pq} , a g_4^1 such that $2g_4^1 \equiv K_X$.

For cubics, being singular is one condition. Hence in a conic of cubics as above there are at least a finite number of singular cubics. Let C" be a singular such cubic. The geometric genus (i.e., the genus of the normalization) of C" is 0 . Let $\tilde{C}$" be the inverse image of C" in B . This has degree 12 because it is the complete intersection of B and the cone over C" with vertex p_1 . As C" is everywhere tangent to the branch locus of B $\to \mathbb{P}^2$, the map $\tilde{C}" \to C"$ is unramified and $\tilde{C}$" breaks into two components C_1'' and C_2'' . Each of these is a space sextic. By [C2] these curves are determinantal and the quartics containing them are determinantal.

Conversely suppose that B is determinantal. Delete one row of the determinant defining B , then the three by three minors that are left define a $\mathbb{P}^2$ of cubics in $\mathbb{P}^3$ whose base locus is a sextic C' of genus 3 on B containing the double points of B . Each of the cubics containing C' cuts, residually on B, another sextic of genus 3 . We see that B contains a $\mathbb{P}^3$ of sextics of genus 3 (see [C2]) which all pass through the double points of B .

By [C2], Θ is parametrized by the set of sextics in Z with a triple point at p_1

and passing through p_i for $i > 1$. Each such sextic C' projects in $\mathbb{P}^2$ to a cubic C" through $p_2,\ldots,p_6$. The lift, determined by C' via incidence, of the canonical divisor cut by C" on X^{pq} gives the image of C' by the Abel-Jacobi map AJ.

Having a triple point at p_1 is two conditions for the $\mathbb{P}^3$ of sextics contained in B. Hence there is a $\mathbb{P}^1$ of such sextics C'. The image of such a C' in $\mathbb{P}^2$ is a cubic C" everywhere tangent to X^{pq} because the inverse image of C" in B has two components.

Notice that, a posteriori, we obtain the fact that if X^{pq} has a conic of everywhere tangent cubics through its double points, then the inverse images of these cubics in B are split.

We have proved :

(10.4) **PROPOSITION** : *The branch loci of the double solids above* $T^* \setminus H_0$ *are determinantal.*

We turn to jacobians. Let Z be a double solid with a unode p_0 (see 1.8) and three ordinary double points p_1, p_2, p_3 (Z generic for these properties). Donagi observes :

(10.5) **PROPOSITION** : JZ *is the jacobian of a smooth curve of genus* 4 .

Proof : Let x, y, z, w be coordinates on $\mathbb{P}^3$ such that $p_0 = (0,0,0,1)$ and near p_0, B has the expansion

$$B = x^2 + (y - az).(y - bz).(y - cz) + \text{higher order} .$$

We can also suppose that $p_1 = (1,0,0,0)$, $p_2 = (0,1,0,0)$, $p_3 = (0,0,1,0)$. The plane representation of X_1 has a triple point (with two other double points) and X_1 is trigonal. This is because if X_1 is given by the equation f locally near p_0, then B has local equation $x^2 - f = 0$ near p_0. The Prym of a trigonal curve is always a jacobian [R].

Q.E.D.

(10.6) Donagi also observes that the discriminant curve X_0 for the projection from p_0

has 6 double points. Three of them are colinear : these are the infinitely near double points p_4, p_5, p_6 in the tangent cone to B at p_0. Normalizing X_0 at five of its double points and keeping p_i $(4 \leq i \leq 6)$one obtains the discriminant curve X_i. These all have the same normalization C and $A = JZ = JC$:

In $\widetilde{\mathbb{P}}^3$ the exceptional quadric $\mathcal{Q}_0$ above p_0 is the union of two projective planes meeting in a line : the tangent cone to B at p_0. Hence, as expected, the double cover of X_0 in $\mathcal{Q}_0$ is split. Its two components are smooth (hence isomorphic to C) and meet in the following way : let p_i be obtained by identifying p_i' and p_i'' on C, then p_i' on one component is identified with p_i'' on the other. So the curves $\widetilde{X}_i$ are of the singular type in 4.2 for $i \geq 4$.

We claim that the only double solids with intermediate jacobian the jacobian of a curve C are the unodal ones or the images of the unodal ones by one of the Cremona transformations described in 1.9 . Let X be a smooth trigonal curve in $P^{-1}(JC)$.

(10.7) **LEMMA** : *Every plane representation of* X *of degree* 6 *has a triple point.*

Proof : Choose a g_6^2 on X. We need to show that $h^0(g_6^2 - g_3^1) > 0$. Let p and q be two points on X such that $h^0(g_6^2 - p - q) = 2$. Put $g_4^1 = g_6^2 - p - q$. The variety W_4^1 of g_4^1's on X is equal to

$$g_3^1 + X \quad \cup \quad K_X - g_3^1 - X$$

(see [W3] or [ACGH] or [AM]). If $g_4^1 = g_3^1 + t_0$ for some t_0 in X, we are done. Otherwise there exists $t_0 \in X$ such that $g_4^1 = K_X - g_3^1 - t_0$.

We have $K_X - g_6^2 = g_3^1 + t_0 - p - q$ is effective by Riemann-Roch. As X is nonhyperelliptic, X has no g_4^2's by Clifford. Two cases are possible :

- $h^0(g_3^1 - p - q) > 0$. Let $p+q+s \in g_3^1$. Then $g_4^1 - s = g_6^2 - p - q - s = g_6^2 - g_3^1$ is effective.

- $t_0 = p$ or q. Say $t_0 = q$. Then $g_6^2 = K_X - g_3^1 + p = g_5^2 + p$. And $g_6^2 - g_3^1 = K_X - 2g_3^1 + p$ is effective because $K_X - 2g_3^1$ is so.

(10.8) **COROLLARY** : *For any plane representation of* X *with an everywhere tangent conic, the double cover of* $\mathbb{P}^2$ *branched along* X *has a unode or is the image of a double solid with a unode by a Cremona transformation.*

Proof : Letting the Cremona group act, we can suppose that a discriminant curve X_i is trigonal. Hence by 10.7, X_i has a triple point in $\mathbb{P}^2$. Then, as in 10.5, it is seen on the local equations for B and X_i that B has a unode.

(10.9) **PROPOSITION** : *If* Z *has two unodes, then* $JC \in \theta_{null} \cap \mathfrak{J}_4$ *, i.e.,* $2\xi = K_C$.

Proof : Suppose that Z has two unodes and is generic for this property. Then the discriminant curve X_0 has one triple point and three ordinary double points which are colinear. Let g^2_6 be associated to this plane representation of C and write $g^2_6 = g^1_3 + t_1 + t_2 + t_3$ (this corresponds to the triple point). Suppose, for instance, that $g^1_3 = \xi$.

Write $K_C - t_1 - t_2 - t_3 = t_4 + t_5 + t_6$. And $\xi - t_i = s_i + u_i$ for $i = 4, 5, 6$. Then

$$g^2_6 - s_i - u_i = t_1 + t_2 + t_3 + t_i = K_C - t_j - t_k \quad \text{with } \{t_i,t_j,t_k\} = \{t_1,t_2,t_3\}$$

is a g^1_4 for all $i = 4, 5, 6$. Thus these give the three double points of X_0. The colinearity condition means that $s_4 + u_4 + s_5 + u_5 + s_6 + u_6 \in g^2_6$. Hence

$$s_5 + u_5 + s_6 + u_6 = g^2_6 - s_4 - u_4 = K_C - t_5 - t_6$$

and

$$K_C = t_5 + s_5 + u_5 + t_6 + s_6 + u_6 = 2\xi .$$

(10.10) We wish to determine the double solids Z which verify $D_Z = t$ the singular point of T.

We have $D_Z = t$ if and only if the lines l_i corresponding to the discriminant curves X_i of Z all pass through t. That is they correspond to elements $\{s,t\} \in C^{(2)}$ (= the minimal desingularization of F) such that $h^0(\xi - s - t) > 0$ or $h^0(K_C - \xi - s - t) > 0$. For instance let $X_1 = C_{st}$ with $h^0(\xi - s - t) > 0$. Then ξ induces a g^1_3 on C_{st}. As C_{st} is in the closure of the locus of smooth trigonal curves, the plane representation X_1 of

C_{st} has a triple point by 10.6 . Hence, as before, the double cover of $\mathbb{P}^2$ branched along X_1 has a unode. The geometric genus of X_0 being 4 , in the most generic case, X_0 has three ordinary double points besides the triple point.

Let $\mathcal{Q}_0$ be the exceptional quadric in $\tilde{Z}$ above the double point p_0 corresponding to C_{st} . $\mathcal{Q}_0$ is the double cover of $\mathbb{P}^2$ branched along C_{0t} . We investigate the condition that the double cover of X_0 in $\mathcal{Q}_0$ is split (see 4.2). Let

$$C_{0t}.X_0 \ = \ 3.(\ \Sigma_{1\leq i\leq 6}\ q_i\ )$$

then, by [C1] (and by continuity because the statement is true generically), the double cover $\tilde{C}_{st} \to C_{st}$ obtained from Z is given by the point of order 2

$$|\ g_6^2\ -\ \Sigma_{1\leq i\leq 6}\ q_i\ |\ .$$

The pull-back of this to C is trivial : $g_6^2 = |\ \Sigma_{1\leq i\leq 6}\ q_i\ |$ (on C) and C_{0t} is twice a line. As C_{0t} is the tangent cone to B at p_0 , the point p_0 is a unode for B (the unode is the simplest singularity for which the tangent cone is twice a line). By 10.9, C has a vanishing theta-null.

The other cases in which all the Prym-curves are singular are specializations of this one. Thus

There are no "double solids" *above* t .

11. TORELLI

Let A be a generic ppav. Let $Z \in \mathcal{Z}_A$ be a double solid with intermediate jacobian A . With the notations of 1.9 , recall that to Z is associated a Kummer variety K with a distinguished double point t_Z . Also recall that if $x \in A$ is such that $\Theta.\Theta_x = E_Z$ is the Fano variety of lines of Z , then the Γ_{00}-map $\tilde{h}$ is ramified at x .

(12.1) **PROPOSITION** : *The (projectivised) kernel of the differential of* $\tilde{h}$ *at* x *is* t_Z *, after we identify* T_xA *with* T_0A *by translation.*

Proof : The kernel of the differential of $\tilde{h}$ at x is the intersection of the tangent spaces at x to the Γ_{00}-divisors which contain x . These divisors are the elements of $H = \tilde{h}(x)$ which is the tangent hyperplane to T at D_Z . Recall that there are 32 double solids with associated Γ_{00}-divisor D_Z , we pick one of these : Z . View D_Z as an element of $(|2\Theta|_{00})^*$, then D_Z is the tangent hyperplane to T* at H . Hence, as T* is in the branch locus of $\tilde{h}$, $D_Z \subset A$ is singular at x .

Now, view D_Z as an element of $|2\Theta|_{00}$, more precisely, of T . The hyperplane H is generated by the six lines l_i in T passing through D_Z . For a generic choice of the 32 double solids above D_Z , as x is a simple ramification point, the kernel of the differential of h at x is one-dimensional. Hence the only Γ_{00}-divisor singular at x is D_Z , and so, given a pencil l_i , the tangent space to the $D \in l_i \setminus \{D_Z\}$ is the same *three-dimensional* vector space. We are going to find the intersection of these.

Let X and X_λ be associated to l_1 . Then $\tilde{X}$ or $\tilde{X}_\lambda$ parametrizes lines in Z through p_1 , suppose, for instance, that $\tilde{X}$ does. Then $\tilde{X}_\lambda$ parametrizes lines in ιZ through p_1 . Suppose that we project Z from p_1 and the plane representation of X is given by $|K_X - s - t|$, then, for a choice of liftings of s and t in $\tilde{X}$, say s', t', we have $x = \pm[s',t'] = s'+t'-\sigma s'-\sigma t'$ (see 2.3). Similarly $x = \pm[u',v']$ where $|K_{X_\lambda} - u - v|$ gives the plane representation of X_λ obtained from Z_λ. These plane representations live naturally in the exceptional plane P_1 above p_1 in $\tilde{\mathbb{P}}^3$. They have the same double points there because their double points are the images of $p_2,\ldots,p_6$ (still denoted by $p_2,\ldots,p_6$). The images of s,t,u,v in P_1 lie on the conic through the points $p_2,\ldots,p_6$ ([C1]). So, in particular, s,t,u,v are not on a line in P_1. Recall that the map ρ is given by the linear system of quadrics through the p_i's. The images of s,t,u,v by the extension of ρ are their Prym-canonical images.

Recall that $x \in \Sigma(X) \cap \Sigma(X_\lambda)$. For every $D \in l_1$, D contains $\Sigma(X) \cup \Sigma(X_\lambda)$. Hence $\mathbb{P}T_x$ contains $\mathbb{P}T_x\Sigma(X) + \mathbb{P}T_x\Sigma(X_\lambda) = \langle s,t\rangle + \langle u,v\rangle \subset \mathbb{P}T_0A$. By the above $\langle s,t\rangle + \langle u,v\rangle$ is a plane and is equal to the image (still denoted by P_1) of P_1 in $\mathbb{P}T_0A$ by the extension of ρ.

Hence the common tangent plane at x to the divisors in l_1 is P_1. And similarly, for all i, the common tangent plane at x to the divisors in l_i is P_i:

$$\begin{array}{ccccc} P_i & \hookrightarrow & \tilde{\mathbb{P}}^3 & \supset & \tilde{B} \\ & \searrow & \downarrow & & \\ & & \mathbb{P}^3 = \mathbb{P}T_0A & & \end{array}$$

By 1.9 the intersection of the P_i's is t_Z. Q.E.D.

12. MODULI OF SIX POINTS

Let A , Z , etc be as above. In $|2\Theta|_{00}$ project from D_Z , this exhibits T as a double cover of $\mathbb{P}^3$ branched along a quartic surface with six double points q_i on a conic $C_{T,D}$: the images of the lines l_i on the image of the tangent cone at D_Z to $T \cap H$.

(12.1) **PROPOSITION** : *The moduli of the points* q_i *on* $C_{T,D}$ *is equal to the moduli of the points* p_i *in* $\mathbb{P}^3$.

Proof: Letting $\mathbb{P}GL(4,\mathbb{C})$ act, we can suppose that $p_1 = q_1$. Project from p_1 , then the q_i's ($i > 1$) project to the five intersection points of Q_X with the image l of $C_{T,D}$. We are reduced to showing that if $q_2,\ldots,q_6$ are five singular quadrics in a pencil l of quadrics containing κX , then the moduli of the q_i in l is equal to the moduli in the conic $C_{s,t}$ containing them (or in $\mathbb{P}^2$) of the double points $p_2,\ldots,p_6$ of the plane representation X^{st} of X given by $|K_X - s - t|$.

Recall that $JX = P(\mathrm{Sing}\Theta',Q)$. The map which to each $q \in Q$ associates its singular point in $|K_X|^*$ is the Prym-canonical embedding χQ of Q [B2]. Let s_i be the singular point of q_i .

The curve X^{st} is also the image of κX by the projection from the line $\langle s,t\rangle \subset |K_X|^*$. Also p_i is the image of s_i by the projection from $\langle s,t\rangle$.

There exists a unique rational normal curve $R_{s,t}$ of degree 4 passing through $s_2,\ldots, s_6, s, t$. Moreover, as the image of $R_{s,t}$ in $|K_X - s - t|^*$ is the conic $C_{s,t}$ which also contains the images of s and t , $R_{s,t}$ is tangent to κX at s and t . Further, the moduli of the p_i's on $C_{s,t}$ is equal to the moduli of the s_i's on $R_{s,t}$.

Now consider the map $\chi : |K_X|^* = \mathbb{P}^4 \rightarrow (\mathbb{P}^2)^* = N^*$ given by the net N of quadrics containing κX . By [B2] the image of χQ by χ is equal to the image of $Q \subset N$ by the dual map of Q . From the above considerations, it is immediately seen that $\chi(R_{s,t})$ is equal to the image of 1 by the dual map of Q and that the s_i's go to the images of the q_i's . Q.E.D.

13. THETA-IDENTITIES

(13.1) Our theta identities are a translation of the scheme theoretic equality

$$E_Z \cap (\Theta_c \cup \Theta_{-c}) = E_Z \cap D_Z$$

(for some $c \in A$ and an appropriate choice of base point for the embedding of the fano variety of lines of $\tilde{Z}$ in A) which we will prove below.

First notice that the 27 (section 4) Prym curves in E_Z are (see 4.8) : $\tilde{X}_i$, $(\tilde{X}_i)_\lambda$ and $\tilde{X}_{ij}$ for all $i \neq j$ between 1 and 6 . We also know the two (see 3.2) translates of each curve inside E_Z :

For the first 12 curves the two copies are obtained by adjoining to the strict transform of a line in Z (resp. $\iota\tilde{Z}$) through p_i one of the two rulings of the exceptional quadric above p_i in $\tilde{Z}$ (resp. $\iota\tilde{Z}$, see 3.5) (to obtain actual lines in $\tilde{Z}$ (or $\iota\tilde{Z}$)) . For the $\tilde{X}_{ij}$, the two copies are obtained by subtracting one of the two lines in Z through p_i and p_j from the sum of each two lines giving us an element of $\tilde{X}_{ij}$ (2.6). In particular, whether we are working with $\tilde{Z}$ or $\iota\tilde{Z}$ we obtain the same Prym-translates of $\tilde{X}_{ij}$ in A (i.e., $\tilde{X}_{ij}$ can also be regarded as the set of incident pairs of twisted cubics).

Inside D_Z we have the surfaces $\Sigma(X_i)$, $\Sigma((X_i)_\lambda)$ and (at least) 4 Prym-translates of $\tilde{X}_{ij}$ because of the choices of ruling above p_i and p_j . Recall (2.1) that the inverse images in $\tilde{Z}$ of lines in $\mathbb{P}^3$ that are tangent to B are blown down to a point by AJ which is the base point for the embedding of D_Z in A .

Taking as base point for the embedding of E_Z in A a line through p_i and p_j in $\tilde{Z}$ we see that five of the curves in E_Z are also in D_Z (this is inspired by [C1]) :

Let l_{ij} be the line in $\mathbb{P}^3$ through p_i, p_j ; l_{ij}', l_{ij}'' be the 2 lines in Z above it and R_i^1, R_i^2 be the rulings of the exceptional quadrics in $\widetilde{Z}$ above p_i . If we choose $R_i^1+l_{ij}'+R_j^1$ as base point for the embedding of E_Z the 5 curves are :

$\widetilde{X}_i+R_i^1$, $(\widetilde{X}_i)_\lambda+R_i^1$, $\widetilde{X}_j+R_j^1$, $(\widetilde{X}_j)_\lambda+R_j^1$, $\widetilde{X}_{ij}$ - l_{ij}'' (= $\widetilde{X}_{ij}+R_i^2+R_j^2$ viewed in D_Z)

Parametrizing Θ as in 2.4 with $(X_{12})_\lambda$ as parameter curve we can suppose that $E_Z = \Theta_a.\Theta_b$ with $b - a = [p,q] \in \Sigma((X_{12})_\lambda)$. With the notations of 2.4 we have for instance $\widetilde{X}_{12} - l_{12}'' = W_p+a$ (cf. 3.2), then, by 5.20, if g_{12} is the g_4^1 relating $(X_{12})_\lambda$ to X_1 and $(X_2)_\lambda$, we have $h^0(g_{12} - \pi p - \pi q) > 0$. Letting $g_{12} \equiv \pi p+\pi q+\pi r+\pi x$, we have by 2.5 that the intersections

$$\Theta \cap \Theta_{[p,q]} \cap \Theta_{[p,r]} \ , \ \Theta \cap \Theta_{[p,q]} \cap \Theta_{[p,\sigma r]}$$

$$\Theta \cap \Theta_{[p,q]} \cap \Theta_{[p,x]} \ , \ \Theta \cap \Theta_{[p,q]} \cap \Theta_{[p,\sigma x]}$$

are unions of W_p and Prym-translates of $\widetilde{X}_1$ and $(\widetilde{X}_2)_\lambda$. Thus these four intersections give us the four possible combinations between 1 translate of $\widetilde{X}_1$ and 1 translate of $(\widetilde{X}_2)_\lambda$ in E_Z , and we have for instance :

$$(\widetilde{X}_1+R_1^1) \cup ((\widetilde{X}_2)_\lambda+R_2^1) \cup (\widetilde{X}_{12} - l_{12}'') \ = \ \Theta_a \cap \Theta_{a+[p,q]} \cap \Theta_{a+[p,r]}$$

Letting $c = a+[p,r]$ we deduce :

$$\Theta_a \cap \Theta_b \cap \Theta_c \subset \Theta_a \cap \Theta_b \cap D_Z$$

and a computation analogous to that of [BD1] (proof of proposition 2) gives an equation with constant coefficients :

(13.2) $$\lambda\theta_a\theta_{-a}+\mu\theta_b\theta_{-b}+\nu\theta_c\theta_{-c}+D_Z \ = \ 0$$

where θ_a is a nonzero section of $\mathcal{O}_A(\Theta_a)$ and we denote D_Z and an equation for it by the same symbol (the coefficient of D_Z is nonzero because A is generic and its Kummer has no trisecants [ref]). From the above equation we deduce :

(13.3) $$\Theta_a \cap \Theta_b \cap (\Theta_c \cup \Theta_{-c}) \ = \ \Theta_a \cap \Theta_b \cap D_Z$$

So $\Theta_a \cap \Theta_b \cap \Theta_{-c}$ contains $(\widetilde{X}_1)_\lambda+R_1^1$ and $\widetilde{X}_2+R_2^1$ and, by 3.2,

$$-c-a \in \Sigma(X_1) \cap \Sigma((X_2)_\lambda)$$

then we deduce from 5.13 that $-c-a = [s,t] \in \Sigma((X_{12})_\lambda)$ and also $h^0(h_{12} - \pi s - \pi t) > 0$

where h_{12} is the opposite g^1_4 of g_{12}. Hence the divisor Θ_{-c} contains a Prym-translate of $\tilde{X}_{12}$ which is also in $\Theta_a \cap \Theta_b \cap D_Z$ and can only be $\tilde{X}_{12} - l_{12}"$.

Now, as before, we also have : $h^0(h_{12} - \pi p - \pi q) > 0$. Putting $h_{12} \equiv \pi p+\pi q+\pi r'+\pi x'$, we obtain (as before) :

$$\Theta_a \cap \Theta_b \cap (\Theta_d \cup \Theta_{-d}) = \Theta_a \cap \Theta_b \cap D_Z$$

where for instance $d = a+[p,r']$ and

$$((\tilde{X}_1)_\lambda + R^1_1) \cup (\tilde{X}_2 + R^1_2) \cup (\tilde{X}_{12} - l_{12}") = \Theta_a \cap \Theta_b \cap \Theta_d$$

Again the same type of computation as in [BD1] (proof of proposition 2) yields that two distinct translates of Θ (different from Θ) have distinct traces on $\Theta_a \cap \Theta_b$ so our only possibility is $d = -c$. We thus obtain the following values for a, b, c :

$$a = 1/2([\sigma p, \sigma p] + [\sigma r, \sigma r']) , \quad b = 1/2([q,q] + [\sigma r, \sigma r']) , \quad c = 1/2[r, \sigma r']$$

with compatible halves.

This takes care of the black triangles in the octahedron (5.30).

(13.4) Our next aim is to write a relation between traces of translates of Θ on E_Z and the incidence divisors D_l on E_Z : for a line l in $\tilde{Z}$, D_l is the family of lines incident to it (see 2.1).

It is proven in [W1] (pp. 77-78) that if we fix a line l_0 in $\tilde{Z}$ then for all l, $D_l - (\Theta_{l_0 - l})|_{E_Z}$ is constant in $\mathrm{Pic}E_Z$. Here Θ can be replaced by any translate of it, so replacing Θ by Θ_{-d} ($d = a+[p,x]$ or $a+[p,\sigma x]$, see above) and taking $l = l_0 = l_{12}'+R^1_1+R^1_2$ we see that this constant is $(\tilde{X}_1+R^2_1) - ((\tilde{X}_1)_\lambda+R^1_1)$ (the set theoretic computation of D_l is easy, we need to see that the components occur with multiplicity 1 :

For a double solid with smooth quartic branch locus we deduce from [C1] and [W1] that the homology class of $D_l \subset E_Z$ in JZ is $1/3(2\Theta^8/8!).\Theta = 6\Theta^9/9!$. Degenerating one double point at a time to the case of a branch locus with six double points (as in 2.1) we see that the homology class of D_l in $A = J\tilde{Z}$ is Θ^3 . So the constant $D_l - (\Theta_{l_0 - l})|_{E_Z}$ is algebraically equivalent to 0 in A .)

The white triangles in the octahedron are incidence divisors on E_Z.

Now by the Seesaw theorem ([M1] p. 54) we can complete the result of [W1] in this case (p. 77) :

If $\phi : E_Z \times E_Z \longrightarrow A$ is the sum and ρ_i the projections $E_Z \times E_Z \longrightarrow E_Z$, then :

$$\phi^*\mathcal{O}_A(\Theta_{-d}) \equiv \mathcal{I} \otimes \rho_1^*\mathcal{O}(\tilde{X}_1-(\tilde{X}_1)_\lambda) \otimes \rho_2^*\mathcal{O}(\tilde{X}_1-(\tilde{X}_1)_\lambda)$$

where $\mathcal{I}$ is the incidence divisor on $E_Z \times E_Z$ and we omit the base points to simplify notations.

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